4 Truth-Taking and Truth-Making

How They Share a Rational Form

Ulf Hlobil

So far, we have pursued a pragmatics-first approach to reason relations. We began with the practice of giving reasons for and against claims. This practice is constrained by reason relations between the contents of claims; and we often let relations among sentences with those contents go proxy for the relations among contents.[1] In the first instance, we understood such reason relations to hold among sentences that do not contain logical vocabulary, that is, among atomic sentences. And we understood logic to be concerned with the norms that govern discourse in two ways: First, we introduced logical vocabulary by specifying the role of the logical lexicon in reason relations, namely by giving sequent rules. Second, we claimed that it is the characteristic job of logical vocabulary to make explicit reason relations, and hence to make explicit constraints on discursive norms governing the use of sentences.

In this chapter, we turn to a semantics-first approach to reason relations. Following such an approach, one aims to give an account of reason relations in terms of relations among what is represented by sentences. Hence, a semantics-first approach to reason relations must start with an account of what and how sentences represent. Our aim in this chapter is to show that there is a way to think about the sentences of our language as representing how things are, in such a way that the account is isomorphic to the pragmatics-first approach presented in the previous chapters.[2] This isomorphism is at the heart of the account of representation that we develop below. And the relation between metavocabularies for reason relations that the isomorphism specifies is at the heart of this book.

The upshot of this chapter is that we can give an account of the reason relations among sentences either in a pragmatic-normative metavocabulary, by specifying the norms that govern the use of the sentences, or we can give an account of these reason relations in a semantic-representationalist metavocabulary, by specifying the modal relations among states represented by sentences. Crucially, however, we can develop these two accounts in such a way that they are isomorphic. We thus arrive at an account of how assertions and denials of sentences—or the acceptance and rejection of claims—represent worldly states that is isomorphic to the account of the norms that govern the uses of sentences. And, as we shall see, this underwrites a version of conceptual realism.

4.1 Rational Forms

In this section, we put in place a semantics-first approach to reason relations at a conceptual level. We codify this approach formally in the truth-maker theory discussed in the following sections.

4.1.1 Representation and Truth

A semantics-first approach to content and reason relations starts with the idea of representation, which is closely related to the idea of truth. What is true or false are contentful items, such as (uses of) sentences, or perhaps mental states or acts.[3] For the sake of simplicity, we will restrict ourselves to sentences as contentful items and, hence, as vehicles of representation. But what we say is meant to apply to other vehicles of representation as well.[4] For our current purposes, a vehicle of representation is anything that can be true or false. So, in keeping with our focus on the sentential level, the kind of representation that is our topic here is not the representation of objects or properties but rather the representation of whatever it is that declarative sentences represent.

What does it mean for a sentence to be true or false? We can understand the truth and falsity of a sentence in terms of the accuracy of uses of the sentence in assertions and denials. To assert that things are thus-and-so while things are thus-and-so or to deny that things are thus-and-so while things are not thus-and-so is accurate. To assert that things are thus-and-so while things are not thus-and-so or to deny that things are thus-and-so while things are thus-and-so is inaccurate. A sentence is true if its assertion is accurate and its denial is inaccurate, and a sentence is false if its assertion is inaccurate and its denial accurate.[5]

A sentence represents things to be a particular way if how these things really stand serves as a standard of truth and falsity for the sentence. Thus, there is a norm governing representations that deems them true or false according to how things stand with what is represented. We may call this essential feature of representation "Normative Governance."

Normative Governance
If ϕ represents x, then whether ϕ is true or false depends on how things stand regarding x.

It follows from Normative Governance that what a sentence represents is, at least in part, a matter of the conditions under which the sentence is true and the conditions under which it is false. There are, however, different ways to spell out this idea of conditions of truth and falsity. Since we are interested in open reason relations and, hence, in frameworks that allow us to reject global structural closure principles, we must be careful to think about conditions of truth and falsity in a fine-grained and flexible way. It will become clear below that we can do so in the framework of truth-maker theory. In truth-maker theory, one thinks about the conditions of truth and falsity of a sentence in terms of states that make a sentence true or false.

Truth-maker theory does not commit one to any particular account of the states that are truth-makers and falsity-makers. They could be events, states of affairs, objects, properties, and more. It is even often convenient to have models in which states are (sets of) sentences. As we shall see below, all that is formally required of states is that they are partially ordered, that every subset of them has a least upper bound with respect to this partial order, and that there is a partition among them into the possible states and the impossible states.

In a philosophical account of representation, however, we should say something about what we take states to be, beyond the very thin requirements of the formalism. To understand what states are, note that a false sentence represents things as being a certain way while that is not how things really are. This way for things to be, while they might actually be otherwise, is a kind of state. It is a kind of state and not a particular state, because there might be many specific ways in which things could be as the sentence represents them to be. The states of this kind are such that the sentence is true if such a state is a part of how things really are, that is, if a state of this kind obtains.[6] These states are the truth-makers of the sentence. Similarly, the states that are such that the sentence is false if they obtain are the falsity-makers of the sentence. To assert a sentence is to purport to state a fact, because it is to convey that one of the states that would make the sentence true is part of how things really are. And to deny a sentence is to purport to state a fact, because it is to convey that one of the states that make the sentence false is part of how things really are.

It is a tricky question what the falsity-makers of many sentences are. What is, for instance, a falsity-maker of “Wittgenstein was French”? Is it perhaps the state of Wittgenstein having been Austrian? Or do we have to posit a negative state of Wittgenstein failing to be French? Or is it the state of reality failing to contain a truth-maker for “Wittgenstein was French”? We will leave such questions unanswered, as all that matters for our purposes is that a falsity-maker is whatever makes the denial of a sentence accurate. It is worth noting, however, that many states leave the truth or falsity of many sentences undecided. Plausibly, the state of Socrates having been Greek, for instance, makes the sentence "Wittgenstein was French" neither true nor false. In fact, our notion of a state is such that the state of Socrates having been Greek does not include anything that is not relevant for whether Socrates was Greek. In Kit Fine's terminology, our truth-makers and falsity-makers are *exact* truth-makers and *exact* falsity-makers. It is also worth noting that we do not exclude impossible states, such as the state of two plus two being five. These are respects in which our notion of conditions of truth and falsity is fine-grained and differs, for instance, from the notion of possible worlds.

At this point, we should distinguish between two senses of "to represent." There is a sense of "to represent" according to which a true sentence represents the obtaining worldly state(s) in virtue of which the sentence is true, and a false sentence represents the obtaining worldly state(s) in virtue of which the sentence is false. Let us call this "factive representation." In this factive sense, only states that obtain can be represented; non-obtaining states cannot be represented. But we can also talk about what a sentence represents without presupposing that it is true or that it is false, thus allowing that the states that we say the sentence represents might not obtain. Let's call this "non-factive representation."[7] It will prove convenient to introduce the label "worldly proposition" for what a sentence represents in this non-factive sense. The worldly proposition that a sentence represents is given by the truth-makers and the falsity-makers of the sentence, and we can think of it, by convention, as the pair whose first member is the set of the sentence's truth-makers and whose second member is the set of its falsity-makers.[8] What a sentence factively represents is contained in what it non-factively represents (either as a truth- or as a falsity-maker) but not vice versa.

Let us sum up what we said so far. We said that a sentence (non-factively) represents the worldly proposition that is the pair of the sentence's truth-makers and the sentence's falsity-makers. This is a notion of representation, because it makes how things really are the standard for the truth or falsity of the sentence, thus underwriting Normative Governance. Moreover, the account connects with the pragmatics of using the sentence because an assertion of the sentence is accurate if and only if the asserted sentence is true, and a denial is accurate if and only if the asserted sentence is false. Note, however, that so far we have not said anything about what makes certain states a sentence's truth-makers and others its falsity-makers. In other words, we have so far not addressed the question: What makes a particular sentence represent a particular worldly proposition? Any representationalist, semantics-first account of reason relations and contents is crucially incomplete and underdeveloped unless and until it answers such metasemantic questions.

The question of what makes some states truth-makers and other falsity-makers of a sentence can be understood in two different ways. According to the first way of understanding the question, it asks what it takes for a sentence to represent a particular worldly proposition: What counts as a sentence representing a worldly proposition? According to the second way of understanding the question, it asks how a sentence acquires and keeps a worldly proposition as that which it represents: How do speakers manage (what do they have to do) to make their sentences (continue to) represent particular worldly propositions?

We will be mainly concerned with the first of these two questions. Before addressing these questions, however, it is worth noting that we can change what a sentence represents by changing the norms that govern the correct use of the sentence. If we all started, for instance, to assess uses of the sentence "Fido is a cat" regarding its truth, justification, entailments, and the like (along with the further changes that this might require, for instance, in assessing uses of other sentences) as we now assess uses of the sentence "Fido is a dog," then the worldly proposition that is represented by "Fido is a cat" would change and would be the worldly proposition that is now (actually) represented by "Fido is a dog." Hence, what makes a particular sentence represent a particular worldly proposition must be a matter of the norms that govern the use of sentences. This is one place where the pragmatics-first approach from the previous chapters and the semantics-first approach that we are pursuing here are linked.

The question what it takes for a sentence to represent a particular worldly proposition thus turns into the following question: What must the norms that govern the use of sentences be like in order for a particular sentence to represent a particular worldly proposition?

The answer that we shall articulate in this chapter is that a sentence represents a worldly proposition just in case the sentence and the worldly proposition share what we call their "rational form."[9] A sentence and a worldly proposition share their rational form if a certain isomorphism holds between uses of the sentence and the states in a worldly proposition. More specifically, the relevant isomorphism must relate assertions of a sentence and its truth-makers as well as denials of a sentence and its falsity-makers, and the isomorphism holds with respect to the modal profile of assertions/denials and truth-/falsity-makers.

The question how a sentence acquires and keeps a worldly proposition as that which it represents, correspondingly, turns into the question: How do the norms that govern the use of sentences emerge and develop so as to make particular sentences represent particular worldly propositions?

The general shape that an answer to this question should take, we think, is that the norms that govern the use of sentences must be sensitive to and shaped by what these sentences represent. We call this idea "Covariant Tracking."

> *Covariant Tracking*
> If y represents x, then y would be different if x had been relevantly different.

The general idea behind Covariant Tracking is that in order for y to represent x, the representation y must tell us something about x. And it is not clear how this could be the case if y would be as it actually is independently of what is the case regarding x.[10] It makes a big difference, however, whether what we mean by "represents" in Covariant Tracking is factive representation or non-factive representation. If we mean factive representation, then the most plausible reading of Covariant Tracking is this: If sentence y factively represents its truth-maker (falsity-maker) x, then agents would not tend to assert (deny) y (to the extent that they do) if x had not obtained. Let us call this "Epistemic Covariant Tracking." If we understand Covariant Tracking as a condition on non-factive representation, however, the most plausible reading of it is very different, namely something like the following: If sentence y represents the worldly proposition x, then the norms governing assertions and denials of y would be different if something relevant about the worldly proposition x had been different. Let's call this "Semantic Covariant Tracking." As we just formulated it, it contains parameters that would need to be specified: What kind of norms governing assertions and denials are at issue? What are the relevant aspects of worldly propositions that would make a difference to these norms? As will become clear below, we hold that the answers to these questions bring in normative and alethic relations of incompatibility.

Epistemic and Semantic Covariant Tracking are related in complex ways. Fortunately, we can avoid these complications for our current project. For our purposes, it is Semantic Covariant Tracking that matters. And what Semantic Covariant Tracking requires is that if a sentence represents a worldly proposition, then the rational form of the sentence would be different if the rational form of the worldly proposition had been different. We will simply assume that this condition holds for the sentences that we are discussing. We, hence, assume that we have a way to adjust the norms that govern our sentences to the rational form of the worldly propositions that we aim to represent with our sentences. However, we leave a discussion of how we do this for another occasion.

To sum up, our view can be described as hylomorphic conceptual realism. The view is hylomorphic because it takes representation to be a relation between hylomorphic compounds that share their form but differ in their matter. The sentence that is representing and the worldly proposition that it represents share their form, namely their rational form. The matter of a sentence consists in tokenings of what Sellars (1963) calls a "sign design" (as assertions or denials), while the matter of a worldly proposition is the matter of worldly states. The rational form is what makes tokenings of sign designs into an occurrence of a particular contentful sentence, and it is also what makes the matter of which a worldly state consists into an occurrence of a particular worldly proposition. The view is a version of conceptual realism because it says that conceptual structure occurs in reality as it is independently of our representation of it. For, to have a particular conceptual structure is to be informed by a particular rational form, and that rational form can be found not just in that which represents but also in that which is represented. When we accurately assert that things are thus-and-so, then what we say and what is the case is the same, in the sense that what we say and what is the case are occurrences of the same rational form, and *mutatis mutandis* for denials. Thus, the conceptual structure, the conceptual *morphe*, is realized in discursive acts and in worldly states.

The remainder of this chapter is an articulation of what it takes for a sentence and a worldly proposition to share a rational form. We will thereby provide a partial explanation of what rational forms are. The answer that we articulate here and defend over the course of the remaining chapters has two key components, namely functionalism and modalism about rational forms. Let us look at these two key components more closely.

4.1.2 Functionalism and Modalism

We maintain that, in virtue of having a particular rational form, that which has the rational form has a particular modal profile. Functionalism and modalism about rational forms concern this modal profile. In order to explain this, let us start with the traditional notion of a substantial form.[11]

The traditional notion of a substantial form is something that can inform different bits of matter, thereby making these bits of matter into unified wholes of a particular kind. Particular substances are what they are in virtue of having the substantial form that they have. Moreover, that something has a certain form has implications about what is possible and impossible for that thing. For example, if two bits of matter share the form of a strawberry, they are, in virtue of having that form, two strawberries, and certain things are possible and others impossible for strawberries. It is, for instance, impossible for a strawberry to be ripe but green, but it is possible for a strawberry to be larger than a grape.

Rational forms also have these traditional features of substantial forms: When a rational form informs some matter, this yields a unity of a specific kind with a particular profile of what is possible and impossible for it. If a rational form informs a tokening of a sign design, then it makes it into an occurrence of a particular contentful sentence. And if a rational form informs a bit of the kind of matter that makes up worldly states, then it makes it into an occurrence of a particular worldly proposition.[12]

Functionalism about rational forms is the thesis that it is necessary and sufficient for something to have a particular rational form that it plays a particular role with respect to other things that have rational forms. Modalism about rational forms is the thesis that the role that something must play in order to have a particular rational form is a role in modal relations. These two theses together yield the claim that what it is to have a particular rational form is to have a particular modal profile with respect to other things that also have rational forms.

The two kinds of bearer of rational forms that are central to our discussion here are sentences and worldly propositions. For we want to know what it takes for a sentence and a worldly proposition to share their rational form. We hold that rational forms give their bearers, that is, the hylomorphic compounds, a very special modal profile. Indeed, to have a particular rational form just is to have a particular modal profile. And in order to understand this modal profile, we must note that bearers of rational forms occur in one of two opposing ways, which we may call "positive" and "negative" occurrences of the rational form. The positive occurrence of a rational form in a sentence is an assertion of the sentence, and a negative occurrence is a denial of the sentence. A positive occurrence of a rational form in a worldly proposition is a truth-maker of the worldly proposition, and a negative occurrence is a falsity-maker of that worldly proposition.

Positive and negative occurrences of rational forms stand in relations of compatibility and incompatibility.[13] What it means for two occurrences of rational forms to be incompatible is that their combination is impossible. The modal profile that bearers of rational forms have consists in the compatibility or incompatibility of their positive and negative occurrences with further occurrences of rational forms, in the same kind of matter. Thus, the modal profile of two bearers of rational forms is identical just in case their positive and negative occurrences are compatible and incompatible with exactly the same further occurrences of bearers of rational forms. That is, the modal profile of two bearers of rational forms are identical just in case the respective combinations of their positive and negative occurrences with other occurrences of rational forms are always either both possible or both impossible.

It will become important below that the kind of impossibility with respect to which a bearer of a rational form plays its role can differ with the matter in which the rational role occurs. In particular, when a rational form occurs in the tokening of a sentence, the kind of impossibility is normative. When the form occurs in a worldly state, however, the relevant kind of impossibility is alethic. This suggests that what it takes for a sentence to have the rational form of a worldly proposition is that assertions of the sentence are normatively incompatible with exactly the assertions (denials) of those sentences whose truth-makers (falsity-makers) are alethically incompatible with the truth-makers of the worldly proposition, and denials of the sentence are normatively incompatible with exactly the assertions (denials) of those sentences whose truth-makers (falsity-makers) are alethically incompatible with the falsity-makers of the worldly proposition.

In order to formulate this idea more precisely, let upper case Latin letters range over sentences. Let $+(\phi)$ stand for an assertion of ϕ, and $-(\phi)$ for its denial. Moreover, let us write $|\phi| = \langle|\phi|^+, |\phi|^-\rangle$ for a worldly proposition that is mapped to sentence ϕ, where $|\phi|^+$ is the set of the sentence's truth-makers and $|\phi|^-$ is the set of its falsity-makers. Then we can formulate our thesis as follows.

> *Sharing Rational Forms*
> The function $|\cdot|$ maps sentences to worldly propositions with which they share their form if and only if, for every sentence S and every two collections of sentences $\{P_1, ..., P_l\}$ and $\{N_1, ..., N_k\}$,
>
> (pos) any combination of $+(S)$ with $+(P_1), ..., +(P_l)$ and $-(N_1), ..., -(N_k)$ is normatively impossible if and only if any combination of states that contains one state from each of $|S|^+$ and $|P_1|^+ ... |P_l|^+$ and $|N_1|^- ... |N_k|^-$ is alethically impossible, and
>
> (neg) any combination of $-(S)$ with $+(P_1), ..., +(P_l)$ and $-(N_1), ..., -(N_k)$ is normatively impossible if and only if any combination of states that contains one state from each of $|S|^-$ and $|P_1|^+ ... |P_l|^+$ and $|N_1|^- ... |N_k|^-$ is alethically impossible.

This can seem rather complex at first, but the idea is simple. For sentences to represent particular worldly propositions, there must be an isomorphism between collections of assertions and denials, on the one side, and collections of truth-makers and falsity-makers, on the other side. In particular, the isomorphism must map assertions of a sentence to truth-makers of the sentence, and it must map denials of the sentence to falsity-makers of the sentence. And this isomorphism must preserve the modal status of the collections it maps to one another. That is, normatively impossible combinations of assertions and denials are mapped to alethically impossible combinations of truth-makers and falsity-makers, and vice versa.

This account can explain why changing the norms that govern the use of a sentence can change the rational form of the sentence. By changing which assertions and denials are normatively incompatible with assertions and denials of a given sentence, we change the rational form of the sentence. Thus, we can make a sentence share the rational form of a worldly proposition and, so, make it represent that proposition by changing the norms that govern assertions and denials of the sentence in the appropriate way.

We can express the appeal to two kinds of modality by saying that our position is a *bimodal* hylomorphic conceptual realism.[14] By this we mean that the conceptual structure of rational forms can inform two kinds of matter that are distinguished by the modal relations that hold among the respective hylomorphic compounds. When rational forms inform matter that occurs in discursive acts, such as tokenings of sentences, then the modality in which occurrences of the forms stand is normative. And when rational forms inform matter that occurs in worldly propositions, such as worldly states, then the modality in which occurrences of forms stand are alethic. Thus, while modal roles define the forms of hylomorphic compounds, it is the matter that decides in what kind of modality these modal roles are realized.

To sum up, functionalism about rational forms says that rational forms are identical just in case their occurrences play the same role with respect to other occurrences of rational forms, in the same kind of matter. And modalism says that to play a particular role, in the sense at issue, is to have a particular modal profile. The two theses together say that what it is to have a particular rational form is to have positive and negative occurrences that are incompatible with certain other occurrences of rational forms. This view is functionalist because it says that what matters for whether something has a particular rational form is not its internal structure but merely the role it plays in a larger whole of similar items. That explains why rational forms can inform very different matter, namely the matter in which represent*ings* occur, such as tokenings of sign designs, and the matter in which represent*eds* occur, like worldly states.

4.1.3 Some Worries about Representation

We said that what it takes for a sentence to represent a particular worldly proposition is that they share their rational form. And we said that what rational form a sentence has must be a matter of the norms that govern its use. Now some noteworthy details of this account of representation start to emerge, and so we may consider some potential objections and replies as a way to further clarify our view.

A sentence represents a worldly proposition just in case they play the same modal role with respect to further sentences and worldly propositions, respectively, and this holds in virtue of appropriate Semantic Covariant Tracking. The sentence and the worldly proposition play the same modal role just in case assertions of the sentence and truth-makers of the worldly proposition as well as denials of the sentence and falsity-makers of the worldly proposition play, respectively, the same modal roles with respect to further assertions/denials and truth-/falsity-makers. This is the claim that we formulated in Sharing-Rational-Forms above.

A first potential problem for any account of representation is that it must make room for misrepresentation. This is easy for our account. Misrepresentation does not occur between sentences and the worldly proposition they represent but between positive and negative occurrences of the sentence and obtaining states in a worldly proposition. An assertion of the sentence misrepresents how things are just in case it is inaccurate, that is, if no truth-maker but rather a falsity-maker of the worldly proposition obtains. And a denial of the sentence misrepresents how things are just in case it is inaccurate, that is, if no falsity-maker but rather a truth-maker of the worldly proposition obtains.[15] By contrast, an assertion represents correctly how things are and, hence, states a fact just in case the assertion is accurate.

A second, and very old, worry about theories of representation according to which what represents must share something with what is represented is that the relation of sharing something is symmetrical, while the relation of representation is not symmetrical. Hence, we must say why a sentence represents a worldly proposition and not vice versa.

Our answer to this second worry is that the asymmetry between representing and represented can be explained in our account by the difference between normative and alethic modality. Occurrences of rational forms where the relevant kind of modality is normative are occurrences that represent, and occurrences of rational forms where the relevant kind of modality is alethic are occurrences that do not represent but can be represented. This is so because the difference between representing and represented is the difference between what is measured and the measure. What is represented is the measure against which the representing is measured: it is the standard for assessments of its correctness. And what is governed by a norm—by a measure—is what is measured. Hence, representings are governed by norms, and in particular the norms pertaining to representational correctness. In contrast to this, representables are not, merely in virtue of being representables, governed by such norms pertaining to representational correctness. The incompatibility relations among them are not normative. There are, of course, complicated cases, such as the representation of representings and alethic modal relations among representings. Such complicated cases are outside of the scope of our foundational project in this book, and we leave their discussion for future occasions. As a response to the current worry, it suffices to say that when rational forms occur in a matter that requires a normative modality as the modality in which they have their characteristic modal profiles, then they inform representings. And when rational forms occur in a matter that requires an alethic modality for their modal profiles, then they inform what can be represented.

A third worry might be that there are too many isomorphisms to uniquely determine a representation relation. We may, for example, envisage a language with just two sentences such that their assertions are incompatible and the denial of either one is compatible with the assertion and with the denial of the other. One might think that these two sentences could represent any number of worldly propositions whose truth-makers are incompatible and whose falsity-makers are compatible, such as the worldly propositions represented in English by "Fido is a dog" and "Fido is a fish" or "This ball is red" and "This ball is green."

Our answer to this worry is that we must distinguish two issues. The first issue is a very general potential problem for any theory of representation. If a structure has some nontrivial automorphism, then any requirement that a certain structure-preserving mapping holds between this structure and another structure does not determine a unique mapping between the two structures. Hence, there cannot be any theory of representation that determines a unique mapping between representings and representeds, that characterizes a structure of representings in such a way that the structure has a nontrivial automorphism, and that characterizes the representation relation by saying that a certain structure-preserving mapping holds between the structure of representings and the structure of representeds.[16] There are three possible ways to address this issue: (a) One might hold that a theory of representation need not determine a unique mapping between representings and representeds. (b) One might try to characterize the structure of representings in a way that does not allow for nontrivial automorphisms. (c) One might include in one's theory of representation characterizations of representings that go beyond the structure of representings as it is in isolation from what is represented. This issue and its three possible approaches to it are not specific to our account of representation, and it seems to us that we are free to adopt any of the three approaches to it.

As philosophers who press the kind of worry that we are currently considering typically prefer an approach to the issue along the lines of (c), let us sketch how this approach could be integrated to our account of representation. Notice that whether an assertion of "This is a dog" is incompatible with an assertion of "This is a fish" depends on whether the speaker points at the same thing while uttering the two tokens of "this." Hence, by our modal functionalism about rational forms, the rational form of "This is a dog" when asserted while pointing at one object is different from the rational form of "This is a dog" when asserted while pointing at another object. So, strictly speaking, we should not talk about the rational forms of sentences but rather about the rational forms of sentences in a particular context of use. In this way, which rational forms occur in what representings may depend on features of the representings that go beyond the structure of representings as it is in isolation from what is represented. We do not commit ourselves to this approach along the lines of (c) here. We merely wish to point out that the issue is a general one, and our account can be fleshed out and extended in ways that is analogous to familiar approaches to the issue.

The second issue is how we should think about potential but nonactual sentences of the language. To see what we mean, consider the example from above again, where we stipulated that our language contains just two sentences whose assertions are incompatible but no other occurrences of them in assertions and denials are incompatible. Are assertions or denials of these sentences incompatible with the assertions or denials of sentences that we might add to our language? If we answer that we meant our stipulation in such a way that the uses of these sentences are compatible with any uses of other sentences that we might add to our language, then what they represent is not uniquely determined only if there are at least two pairs of sets of states (worldly propositions) such that any combination of two states such that one comes from the first member of the first pair and the second from the first member of the second pair are incompatible and the states are otherwise compatible with all other states. The example of the worldly propositions expressed, in English, by "Fido is a dog" and "Fido is a fish" and again by "This ball is red" and "This ball is green" is not an example of such a case. Rather, the structure of alethic incompatibility (and not just the structure of normative incompatibility) has nontrivial automorphisms in such a case, and we will come back to this momentarily. If, however, we answer that we meant our stipulation in such a way that the uses of our two sentences may or may not be compatible with the uses of other sentences that we might add to our language, then the question arises whether such an extension of our language must preserve the isomorphism between normative and alethic incompatibility. We hold that the isomorphism must indeed be preserved by all such possible extensions in order for a representation relation to hold. That is how our account of representation ought to be understood. Hence, in order to determine the isomorphism that explains representation, we must look at all possible extensions of a given language. If two distinct isomorphisms between normative and alethic incompatibilities remain even when we take arbitrary extensions of our language into account, then there must be a nontrivial automorphism in the structure of alethic incompatibilities. Therefore, such an automorphism in the structure of alethic incompatibilities must exist for either one of our two answers to yield a potentially problematic underdetermination of the representation relation. Such an automorphism would be an extremely interesting symmetry in the modal structure of reality. If there is such a symmetry, then there is a corresponding symmetry in the normative incompatibilities. And this automorphism in the normative incompatibilities is responsible for the fact that our account cannot pick out a unique worldly proposition for each sentence as what it (non-factively) represents. In such a case, we are open to the possibility that there is no fact of the matter which of the two isomorphisms constitutes a representation relation. It is difficult to find any realistic example of such an automorphism. Perhaps the automorphism between the mathematics of complex numbers and that same structure with i and $-i$ replaced for each other is an example. If so, we would be open to the possibility that it is indeterminate whether our sentence "$i^2 = -1$" represents the worldly proposition whose truth-maker is that the square of i is -1 or the worldly proposition whose truth-maker is that the square of $-i$ is -1. This seems to us to be a kind of underdetermination of the representation relation that may be acceptable, should it really turn out to be intelligible.

As a final remark regarding the third worry, we want to point out that there may be cases of genuine indeterminacy of which worldly proposition is nonfactively represented by a sentence. Perhaps there is, for instance, no fact of the matter which worldly proposition is represented by the sentence "The mass of the ball is one kilogram" in a Newtonian language. If so, there is some truth in approach (a) to the general issue that a theory may not uniquely determine representation relations, namely the approach that says that such a unique determination is not necessary in order for a theory of representation to be satisfactory. This truth applies more widely than just in the cases mentioned above. So, the fact that our account of representation does not always yield a unique mapping between representings and representeds may ultimately be an advantage rather than a problem.

To sum up, our account of representation can easily handle misrepresentation, it can account for the asymmetry of the representation relation (despite taking representation to be a matter of sameness of form), and it has the resources to address the potential problem that there may be several distinct isomorphisms between normative and alethic incompatibilities.

4.1.4 Reason Relations for Representationalists

According to a semantics-first approach to reason relations, we should explain reason relations among sentences in terms of what these sentences represent. What could such an account look like?

In the previous chapters, we offered an account of reason relations in a normative-pragmatic metavocabulary. According to this account, a sentence is a reason for another sentence if asserting the first and denying the second is normatively impossible (or out-of-bounds, in bilateralist terminology) because one cannot be entitled to such a combination of discursive acts. A sentence is a reason against another sentence if asserting both sentences is normatively forbidden. Given the isomorphism that underwrites the sharing of rational forms, we can give a parallel account of reason relations in terms of alethic modal relations among states, namely the following: A sentence is a reason for another sentence if and only if the combination of any truth-maker of the worldly proposition represented by the first sentence and any falsity-maker of the worldly proposition represented by the second sentence is impossible, in the alethic modal sense. And a sentence is a reason against another sentence if and only if the combination of truth-makers of the two worldly propositions represented by the sentences is alethically impossible.

On this view, a sentence is a reason for another sentence if the truth of the first rules out the falsity of the second. And it is a reason against the other sentence if they cannot be true together. Such impossibilities hold because of the modal relations among the states that make up the worldly propositions that the sentences represent. Thus, reason relations among sentences are explained in terms of relations among what they represent.

At this point, a reminder of what we mean by a sentence being a reason for or against another sentence is in order. As in the previous chapters, the reason relations at issue here do not determine but merely constrain the "reasons" (in the more common sense) we have for particular beliefs in particular circumstances. We are defining reason relations of implication and incompatibility, which are at one remove from practical policies. That one sentence is a reason for another does not imply that you ought to believe what the second sentence says if you believe what the first sentence says. Rather, when you believe what a sentence says on the grounds of what another sentence says, then you believe what the first sentence says for good reason only if the second sentence is a reason for the first sentence, in the sense at issue in this book. And the same holds *mutatis mutandis* for reasons against something.

4.2 Truth-Maker Theory

In the first section of this chapter, we have outlined in very abstract terms the strategy of a particular semantics-first approach to content and reason relations. We have already seen that the idea of representation seems to lead us back to the pragmatics-first approach from the previous chapters. In particular we seem compelled to talk about the norms that govern the use of sentences when we want to say why a particular sentence represents what it represents. In order to make such connections precise, however, we need a formal codification of the representational account sketched above. Fortunately, there is an extant formal theory that can serve this purpose with only minor adjustments, namely the exact truth-maker theory developed by Kit Fine (2017a; 2017b). The goal of this section is to put in place a version of truth-maker theory, which we will later show to be isomorphic to the theory that we have put forward in the previous chapters.

Due to this isomorphism, the ability of the sequent calculus NMMS from the previous chapter to codify open reason relations will extend to this version of truth-maker theory. Thus, the truth-maker semantic theory as amended below is substructural in the same way in which the sequent calculus NMMS is substructural. In order to meet our desideratum of being able to codify open reason relations, we will start with a structural version of truth-maker theory and then show how we can drop the constraints that prevent the closure-structured version from codifying open reason relations.

Truth-maker theory is perhaps the most powerful and flexible version of the broad idea of truth-conditional semantics. Truth-maker theory has been used in metaphysics, for instance, to give accounts of essence (Zylstra, 2019; Hale, 2020), grounding (deRosset and Fine, 2022), and generalized identity (Elgin, 2021). It has been applied in natural language semantics, for instance, to give accounts of attitude reports and deontic modals (Moltmann, 2020), conditionals (Yablo, 2016; Santorio, 2018), and to give accounts of the subject matter of sentences (Yablo, 2014; Fine, 2020). Moreover, truth-maker theory has been used to give accounts of various normative phenomena (Fine, 2018c, a, b), verisimilitude (Fine, 2019), truth (Jago, 2018), and other phenomena. One reason why accounts of all these different and philosophically puzzling phenomena can be formulated in truth-maker theory is that it is hyperintensional. That is, we can distinguish between the contents of sentences whose truth-values are necessarily the same, including sentences that are equivalent according to classical logic. Roughly, the idea is that, say, "$1 + 1 = 2$" and "π is irrational" might differ in content, although they are both necessarily true, because they are made true by different aspects of reality, or states. Similarly, "It is raining" and "It is raining and there is or isn't extraterrestrial life" differ in content, although they are equivalent according to classical logic, because there are aspects of reality, or states, that are relevant to the truth of the second but that are not relevant to the truth of the first, namely how things stand with extraterrestrial life. Moreover, truth-maker theory has the resources to assign distinct semantic values to two necessarily false sentences, such as "It is day and also night" and "The ball is blue all over and yellow all over," namely by allowing for distinct impossible states, such as a state of it being day and night and a distinct state of the ball being blue all over and yellow all over. Thus, we can make very fine-grained distinctions in truth-maker theory, and it is in this sense much more expressively powerful than, for instance, possible world semantics. This is part of what makes this framework flexible enough to allow us to formulate an account of reason relations that is isomorphic to the one from the previous chapters.

Let us formulate in more detail the precise version of truth-maker theory that is relevant for our project. The starting point of truth-maker theory is the idea that there are states that can obtain or not obtain and that make some sentences true, that is, verify them, and make some sentences false, that is, falsify them (Fine, 2017a, 2014). The verifying states (truth-makers) and falsifying states (falsity-makers) of a sentence are wholly relevant to the sentence. The state of it raining, for instance, is a truth-maker of the sentence "It is raining." But the state of it raining and it being cold is not a truth-maker of "It is raining" because it is not wholly relevant to the truth of the sentence (it's being cold is irrelevant). States can be parts of other states. Some states are possible and others are impossible.

Let us follow Fine (2017a, 647) and say that a "modalized state space" is a triple of a set of such states, the subset of those states that are possible states, and the parthood relation among these states:

Definition 35 (Modalized state space)**.** A modalized state space is a triple, $\langle S, S^{\diamond}, \sqsubseteq \rangle$, such that S is a non-empty set of states, $S^{\diamond} \subseteq S$ (intuitively: the possible states), and $\sqsubseteq$ is a partial order on S (intuitively: parthood), such that all subsets of S have a least upper bound.

States can be combined—or, as we shall say, "fused"—to yield further states. We define the fusion of states as the smallest state of which all the initial states are parts, that is, as the least upper bound of the initial set of states under our partial order (Fine, 2017a, 646).

Definition 36 (Fusion)**.** The fusion of a set of states $T = \{t_1, t_2, t_3, ...\}$, written $t_1 \Cup t_2 \Cup t_3 ...$ or $\Cup T$, is the least upper bound of T with respect to $\sqsubseteq$.[17]

Every modalized state space has a least element, $\square$, which is the "null state" that is part of every state. It is the least upper bound of the empty set.

We said above that a sentence, ϕ, of language $\mathfrak{L}$ represents a worldly proposition, which is a pair, $\langle|\phi|^{+}, |\phi|^{-}\rangle$, of the set of the sentence's verifiers, $|\phi|^{+}$, and the set of its falsifiers, $|\phi|^{-}$. We call a mapping of sentences to worldly propositions an interpretation function. A model is a modalized state space together with an interpretation function.

Definition 37 (Model)**.** Given a language $\mathfrak{L}$, a model, $\mathcal{M}$, is a quadruple $\langle S, S^{\diamond}, \sqsubseteq, |\cdot|\rangle$, where $\langle S, S^{\diamond}, \sqsubseteq\rangle$ is a modalized state space and $|\cdot|$ is a interpretation function, such that $|A| = \left\langle |A|^{+}, |A|^{-} \right\rangle \in \mathcal{P}(S) \times \mathcal{P}(S)$.

We write $\mathcal{M}, s \Vdash A$ if a state s verifies a sentence A in model $\mathcal{M}$, and if no risk of confusion arises, simply $s \Vdash A$. Similarly, we write $\mathcal{M}, s \dashv\!| A$ to say that s is a falsifier of sentence A in model $\mathcal{M}$. In other words, relative to a given model, $|A|^{+}$ is the same as $\{s : s \Vdash A\}$, and $|A|^{-}$ is the same as $\{s : s \dashv\!| A\}$.

In order to give a treatment of logical vocabulary in terms of these models, we have to treat some bits of vocabulary as logical vocabulary by keeping its meaning fixed in all models. So let $\mathfrak{L}$ be a language that results from adding $\neg$ and $\wedge$ and $\vee$ to a stock of atomic sentences $\mathfrak{L}_{\mathfrak{B}}$. To hold the meanings of logical vocabulary fixed, we give semantic clauses that specify the truth-makers and falsity-makers of complex sentences in terms of the truth-makers and falsity-makers of their parts.[18] We will limit ourselves throughout to propositional logic, as we did in the previous chapters.[19] For the atomic sentences, our models directly specify their truth-makers and falsity-makers. Thus, we have:

(atom+) $s \Vdash p$ if and only if $s \in |p|^{+}$

(atom−) $s \dashv\!| p$ if and only if $s \in |p|^{-}$

For conjunction, it is plausible that the truth-makers of conjunctions are fusions of the truth-makers of their conjuncts. For example, the sentence "It is raining and it is night" is made true by states that combine a truth-maker of "It is raining" and a truth-maker of "It is night" (and nothing else). And it also seems plausible that a conjunction is made false by any state that makes one of the conjuncts false, or a fusion of such states. For example, the state of it being sunny and the state of it being daytime are intuitively both falsity-makers for "It is raining and it is night." Thus, we have the following semantic clauses for conjunction[20]:

(and+) $s \Vdash B \wedge C$ if and only if
$\exists u, t\, (u \Vdash B$ and $t \Vdash C$ and $s = u \Cup t)$

(and−) $s \dashv\mid B \wedge C$ if and only if
$s \dashv\mid B$ or $s \dashv\mid C$ or $\exists u, t\, (u \dashv\mid B$ and $t \dashv\mid C$ and $s = u \Cup t)$

Disjunction works, plausibly, in an analogous way, with truth-making flipped to falsity-making and vice versa: A state makes a disjunction true if it makes one of the disjuncts true or is a fusion of such states. And this suggests the following clause for the truth-makers of disjunctions.

(or+) $s \Vdash B \vee C$ if and only if
$s \Vdash B$ or $s \Vdash C$ or $\exists u, t\, (u \Vdash B$ and $t \Vdash C$ and $s = u \Cup t)$

Moreover, a state makes a disjunction false if it makes both disjuncts false.

(or−) $s \dashv\mid B \vee C$ if and only if
$\exists u, t\, (u \dashv\mid B$ and $t \dashv\mid C$ and $s = u \Cup t)$

And for negation, it is plausible to assume that a state makes a negation true if it makes the negatum false, and it makes a negation false if it makes the negatum true.

(neg+) $s \Vdash \neg B$ if and only if $s \dashv\mid B$

(neg−) $s \dashv\mid \neg B$ if and only if $s \Vdash B$

It will prove convenient below not only to talk of individual sentences being made true or false, but also of sets of sentences being made true or false. We will say that a set of sentences is made true by the truth-makers of the conjunctions of its members, unless there are no such truth-makers, in which case it is made true by the null state. Similarly, we say that a set of sentences is made false by the falsity-makers of the disjunction of its members, unless there are no such falsity-makers.[21]

Definition 38 (Truth- and falsity-makers of sets). $u \Vdash \Gamma$ if and only if $u \Vdash \bigwedge \Gamma$, unless $\{x : x \Vdash \bigwedge \Gamma\} = \emptyset$ in which case $\square$ and nothing else makes Γ true. And $t \dashv\mid \Delta$ if and only if $t \dashv\mid \bigvee \Delta$, unless $\{x : x \dashv\mid \bigvee \Delta\} = \emptyset$ in which case $\square$ and nothing else makes Δ false.

Given the clauses for conjunction and disjunction, it is easy to see that the truth-makers of sets are the fusions of the truth-makers of their members, if there are any. And the falsity-makers of sets are the fusions of the falsity-makers of their members, if there are any. So the appeal to conjunction and disjunction could be eliminated; it is merely an efficient way to talk about fusions. The unless-clauses are useful because they imply that the empty set is made true and false by the null state. This might seem strange but it has technical advantages. In particular, it allows us to say that fusing a state with the truth-maker or falsity-maker of the empty set simply returns the original state; for, $s \Cup \square = s$.

It will also occasionally be useful to allow ourselves to ignore certain cardinality issues. To see what we mean, note that, given a countable infinity of states, there will be an uncountable infinity of worldly propositions. Hence, we cannot express all worldly propositions in a language. It is sometimes useful to ignore this complication, and we can do so by making the following assumption.[22]

Assumption 39 (*Expressibility of worldly propositions*)**.** For any worldly proposition $\langle V, F \rangle$, we can add sentences, $\Gamma \cup \Delta$, to our language such that V contains exactly one truth-maker for each sentence in Γ (and nothing else) and F contains exactly one falsity-maker for each sentence in Δ (and nothing else).

With this assumption in place, if we encounter a worldly proposition, we can assume that our language has (or can be given) the resources to express the proposition, and to do so in a sufficiently fine-grained way, namely as fine-grained as the states in the proposition. As will become clear in the next two chapters, the assumption is best understood as the mirror image of the fact that our grasp of states is limited by our language.

We have now spelled out most of the details of our models. However, one crucial element is missing. So far, we have put no constraints on the distinguished set of possible states; they might be any subset of our states. Fine often imposes the following constraints, and they will become important below:

Downward-Closure: If $s \in S^\diamond$ and $t \sqsubseteq s$, then $t \in S^\diamond$.

Exclusivity: If $s \in |p|^+$ and $t \in |p|^-$, then $\forall u(s \Cup t \Cup u \notin S^\diamond)$.[23]

Exhaustivity: $\forall u \in S^\diamond$, either $\exists s \in |p|^+ \ (u \Cup s \in S^\diamond)$ or $\exists t \in |p|^- \ (u \Cup t \in S^\diamond)$.

Downward-Closure says that all parts of a possible state are possible. Exclusivity says that if you take any atomic[24] sentence, then if you fuse one of its verifiers with one of its falsifiers together with any other state (or, in effect, none, if it is the null state), you always get an impossible state. And Exhaustivity says that if you have a possible state and an atomic sentence, then you can extend it to a possible state either by fusing it with a verifier of the sentence or by fusing it with a falsifier of the sentence.

It will emerge in due course that the three constraints of Downward-Closure, Exclusivity, and Exhaustivity correspond, respectively, to the structural rules of Weakening, Containment, and Cut in a classical sequent calculus. Moreover, the semantic clauses correspond to our operational sequent rules. This correspondence is at the center of the isomorphism we will establish between the theory from the previous chapters and truth-maker theory.

Before we get to this correspondence, however, we must address what is perhaps the central question in formulating any logical system: How should we define consequence? (For us, this shows up as the key *semantic* question about a logic.) It is common in the literature to define consequence in truth-maker theory in non-modal terms, that is, without any appeal to the distinction between possible and impossible states. In particular, Fine often uses the following notions of consequence[25]:

- *Entailment*: P entails Q if and only if every verifier of P is a verifier of Q (Fine, 2017a, 640–41).
- *Containment*: Q contains P if and only if (i) every verifier of Q includes as a part a verifier of P and (ii) every verifier of P is included as a part in a verifier of Q (Fine, 2017a, 640–41).

These notions can be extended to the case of multiple premises in different ways (Fine and Jago, 2019). As we will see below, however, an interesting equivalence emerges between truth-maker theory and the theory in the previous chapters if we adopt, instead, a different notion of consequence, which we call "truth-maker consequence" or "TM-validity"[26] and write as $\models_{TM}$. This notion captures, within truth-maker theory, the representationalist account of reason relations sketched in the previous section.

Recall that we suggested above that, from a representationalist perspective, what it means for A to be a reason for B is that it is impossible that A is true and B is false. And what it means for A to be a reason against B is that it is impossible that A is true and B is also true. We can now formulate this idea in truth-maker theory as follows: A is a reason for B if and only if any fusion of a truth-maker of A and a falsity-maker of B is an impossible state. And A is a reason against B if and only if any fusion of a truth-maker of A and a truth-maker of B is an impossible state. As in previous chapters, we can think of these relations as codified in a consequence relation by saying that B is a consequence of A if and only if A is a reason for B, and the empty set is a consequence of A together with B if and only if A is a reason against B. Notice that this truth-maker conception of consequence, unlike Fine's candidates, is a *modal* notion; it crucially involves the distinction between possible and impossible states.

According to this notion, *B* is a consequence of *A* just in case no truth-maker of *A* is compossible with any falsity-maker of *B*.[27] If we finally also generalize these intuitive ideas to sets of sentences, the result is the following:

> *Truth-Maker Consequence*: $\Gamma \models_{TM} \Delta$ (in a model) if and only if (in that model) any fusion of verifiers for each $\gamma \in \Gamma$ and falsifiers for each $\delta \in \Delta$ is an impossible state, that is, $s \notin S^{\diamond}$ for all $s = u \Cup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\Vert \Delta$.[28]

This notion of consequence is essentially both modal and bilateralist, in that it makes crucial use of the distinction between possible and impossible states and, moreover, makes crucial use of falsity-makers as well as truth-makers. To the best of our knowledge, these features distinguish this notion of consequence from all extant notions of consequence used in truth-maker theory.[29] As we shall see below, this allows us to make full use of the many fine-grained features of truth-maker theory in thinking about consequence. In particular, it will allow us to capture open reason relations.

It will prove useful in the next chapter to add a definition of an implication relation among worldly propositions, that is, a kind of consequence relation that does not depend on any particular language. We can apply the idea behind Truth-Maker Consequence directly to worldly propositions by saying that a set of worldly propositions implies another set of worldly propositions just in case any fusion of truth-makers for each proposition in the first set and falsity-makers for each proposition in the second set is an impossible state.

> *Propositional Implication*: With *G* and *D* being sets of worldly propositions, $G \Vdash_{PI} D$ (in a modalized state space) if and only if (in that modalized state space) any fusion of verifiers for each $\langle V, F\rangle \in G$ and falsifiers for each $\langle V, F\rangle \in D$ is an impossible state.

We could think of the worldly propositions that would be the interpretation of a negation of a sentence as the (propositional) negation of the proposition that is the interpretation of the negated sentence, and similarly for the other connectives. For example, we can think of $\langle |\phi|^{-}, |\phi|^{+}\rangle$ as the (propositional) negation of $\langle |\phi|^{+}, |\phi|^{-}\rangle$. While this way of thinking will become relevant in the next chapter, however, it is more illuminating for the purposes of the current chapter to think of consequence as a relation among sets of sentences of a language.

Truth-Maker Consequence gives us a notion of consequence relative to models. And if we have a model that models reality (including modal aspects of reality) in the right way, it will be consequence relative to that model that captures what follows from what in the right way. In specifying a particular model, we can choose which material inferences are TM-valid in that model. If we want to ensure, for instance, that "*o* is colored" follows from "*o* is green," all we have to do is to ensure that every fusion of a falsity-maker for "*o* is colored" and a truth-maker for "*o* is green" is an impossible state.[30] Thus, truth-maker consequence is not logical consequence; it is not closed under uniform substitution of non-logical expressions. We are, of course, especially interested in these wider notions of consequence. We can, however, easily restrict truth-maker consequence to logical consequence (LTM-validity) by quantifying over models in the usual way.

> *Logical Truth-Maker Consequence*: $\Gamma \models_{LTM} \Delta$ if and only if, in all models, $s \notin S^{\Diamond}$ for all $s = u \Cup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\mid \Delta$.

As we will see shortly, making different choices in the semantic set-up leads to different consequence relations for TM-validity and LTM-validity. Hence, these are really families of consequence relations. We will disambiguate where necessary by using appropriate labels. It will turn out that different members of these families correspond exactly to the different accounts of consequence from the previous chapter.

4.3 Articulating the Isomorphism

Let us now bring out how the version of truth-maker theory maps onto our sequent calculus NMMS from the previous chapter. As we will see, the left-rules of NMMS turn out to correspond to the semantic clauses for truth-makers. The right-rules turn out to correspond to the clauses for falsity-makers. And the structural rules of Containment, Monotonicity, and Cut will turn out to correspond to constraints on possible states, namely Exclusivity, Downward Closure, and Exhaustivity, respectively. In this section, we will explain these correspondences and some of their implications.[31] In order to do so, we start by explaining the general idea or strategy behind the isomorphism. We then turn to the operational sequent rules, and we end by discussing the structural rules.

4.3.1 The Idea Behind the Isomorphism

Recall that according to the pragmatics-first approach that we have presented in the previous chapters, what it means to say that Δ follows from Γ is that if one is committed to assert everything in Γ, then one cannot be entitled to deny everything in Δ. Equivalently, we said that what it means that Δ follows from Γ is that any combination of assertions of every sentence in Γ and denial of every sentence in Δ is normatively impossible (forbidden) or "out-of-bounds." Let us denote this notion of consequence by $\models_{PN}$, for Pragmatic-Normative consequence. This pragmatic-normative understanding of consequence offers an intuitive interpretation of sequents and sequent rules. In particular, sequent rules can then be interpreted as telling us that certain combinations of assertions and denials are normatively impossible if other such combinations are normatively impossible.

In formulating Sharing-Rational-Forms above, we explained that a sentence and a worldly proposition share their rational forms if their occurrences stand in isomorphic modal relations to other occurrences of sentences and worldly propositions respectively. On the side of sentences, the relevant modality is normative and the occurrences are assertions and denials. And on the side of worldly propositions, the relevant modality is alethic and the occurrences are truth-makers and falsity-makers.

We can now see what it would mean to establish the required isomorphism between sentences and worldly propositions in terms of pragmatic-normative consequence and truth-maker consequence. There would have to be an interpretation that maps sentences to worldly propositions in such a way that pragmatic-normative consequence and truth-maker consequence coincide. We can now formulate this idea as follows:

Sharing-Rational-Forms (formal)
A sentence, A, of a language and the worldly proposition, $|A|$, that is its interpretation (in a model) share their rational form if and only if for every two collections of sentences Γ and Δ,
(pos) $\Gamma, A \models_{PN} \Delta$ if and only if $\Gamma, A \models_{TM} \Delta$
(neg) $\Gamma \models_{PN} A, \Delta$ if and only if $\Gamma \models_{TM} A, \Delta$.

This says that a sentence A shares its rational form with its interpretation $|A|$ (in a model) just in case an assertion of A is normatively incompatible with a collection of assertions and denials if and only if (in that model) every fusion of a truth-maker of A with truth-makers for the assertions and falsity-makers for the denials is an impossible state. And *mutatis mutandis* for denials of A. Thus, this is really just another formulation of Sharing-Rational-Forms from Section 4.1 above.

We have interpreted sequent calculi in a pragmatic-normative way in the previous chapters. And we have introduced truth-maker consequence in the previous section. So, we can now establish the desired isomorphism by mapping truth-maker consequence into our sequent calculus. Crucially, however, this mapping is not just global but also local, that is, we will map the details of the two frameworks to each other, and this will allow us to tweak both frameworks in exactly parallel ways, including the rejection of global closure-structural principles.

To see how this mapping of details works, notice the common structure of pragmatic-normative consequence and truth-maker consequence: First, we have items with positive and negative occurrences. Second, we have a notion of combining these elements into larger structures. Third, we distinguish among such combinations between the possible and the impossible ones. We can read sequent rules as concerning these three elements, and we can, thus, generate two parallel readings of every sequent rule, one pragmatic-normative reading and one truth-maker reading. Take, for instance, the following version of the structural rule of Cut.

$$\frac{\Gamma \succ \Delta, A \qquad A, \Gamma \succ \Delta}{\Gamma \succ \Delta}\ [\text{Cut}]$$

Reading this rule contrapositively, it says that, for any Γ and Δ, if the sequent $\Gamma \succ \Delta$ doesn't hold, then either $\Gamma \succ \Delta, A$ or $A, \Gamma \succ \Delta$ doesn't hold. We can formulate the pragmatic-normative (PN) and the truth-maker bilateralist (TM) readings of this claim as follows:

PN-Cut: For any normatively possible combinations of assertions and denials and any sentence, *A*, one can extend the combination while keeping it normatively possible either by asserting or by denying *A*.

TM-Cut: For any alethically possible state and any worldly proposition, *A*, one can extend the state into an alethically possible state by fusing it with either a verifier or a falsifier of *A*.

Notice that TM-Cut is a version of Fine's Exhaustivity, from the previous section. The only two apparent differences are that Exhaustivity is formulated without mentioning worldly propositions and that it concerns only atomic sentences. Both differences are, however, insubstantial. Below we will spell this equivalence of Exhaustivity and Cut out in a more rigorous fashion.

An analogous result holds for all of the other rules. To see this, let's consider next the principle of Containment, which says that every sequent of the form $\Gamma, A \succ A, \Delta$ holds. The two readings are the following:

PN-CO: Any combination of assertions and denials in which any sentence is asserted and also denied is normatively impossible.

TM-CO: Any state that includes a verifier and also a falsifier for any worldly proposition is alethically impossible.

Notice that TM-CO is a variant of Exclusivity from the previous section. The only and irrelevant difference is that Exclusivity is formulated only for atomic sentences.

The third constraint on possible states, which is Downward Closure, is equivalent to the structural rule of [Weakening] or Monotonicity, which says that if $\Gamma \succ \Delta$ holds, then so does $\Gamma, \Theta \succ \Delta, \Sigma$. The corresponding two readings are these:

PN-MO: All combinations of assertions and denials that include a normatively impossible combination as a part are themselves normatively impossible.

TM-MO: All states that include an alethically impossible state are themselves alethically impossible.

We can also give two interpretations of each of our operational rules. The left-rules and the right-rules all work generically in the same way. Hence, we can formulate the equivalence at this generic level. For the left-rules, we can provide parallel interpretations in the following way:

PN-left: Left-rules specify the contributions that the assertions of complex sentences make to collections of assertions and denials being normatively impossible in terms of the contributions made by the assertions or denials of their constituent sentences.

TM-left: Left-rules specify the contributions that the verifiers of the worldly proposition represented by complex sentences make to states being alethically impossible in terms of the contributions made by the verifiers or falsifiers of their constituent sentences.

And the parallel formulations for the right-rules are as follows:

PN-right: Right-rules specify the contributions that the denials of complex sentences make to collections of assertions and denials being normatively impossible in terms of the contributions made by the assertions or denials of their constituent sentences.

TM-right: Right-rules specify the contributions that the falsifiers of the worldly proposition represented by complex sentences make to states being impossible in terms of the contributions made by the verifiers or falsifiers of their constituent sentences.

We thus have two parallel readings of valid sequents and sequent rules: one alethic-modal, the other pragmatic-normative. In both interpretations, the consequence relation holds between two sets just in case certain states or combinations of assertions and denials are impossible. This impossibility is normative in one case and alethic in the other case.

According to the account of representation from the first section of this chapter, a sentence and a worldly proposition share their rational form just in case the incompatibilities of assertions and denials of the sentence with other assertions and denials are isomorphic to the incompatibilities of the truth-makers and falsity-makers of the worldly proposition with other states. In order for a sentence to represent a worldly proposition, the two must share their rational form.[32] So if we want our sentences to represent particular worldly propositions, we must make it the case that their assertions and denials are governed by norms that are isomorphic in this way. The two readings of sequent rules just sketched means that if such an isomorphism holds for our atomic sentences, then it will also hold for complex sentences, given that we use corresponding sequent rules and truth-maker clauses for our logical vocabulary.

4.3.2 Operational Rules

In this and the following subsection we want to spell out the isomorphism just sketched with more formal rigor. We start with the operational rules in this subsection and turn to the structural rules in the next subsection.

Before we discuss particular sequent rules in detail, however, it will prove useful to introduce the notion of a sequent "deeming certain states impossible." The idea is that a sequent deems exactly those states impossible that need to be impossible in order for the sequent to hold, that is, for the conclusions to be a consequence of the premises. Hence, a sequent deems impossible exactly those states that are fusions of truth-makers of the premises and falsity-makers of the conclusions.

Definition 40 (Deeming impossible). A sequent $\Gamma \succ \Delta$ deems all and only those states impossible that are a fusion of verifiers for everything in Γ and falsifiers for everything in Δ, that is, any state $s = u \Cup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\vert\vert\, \Delta$.

This notion will prove useful because it turns out that, in all our sequent rules, the set of states that are deemed impossible by the bottom-sequent is the union of the sets of states deemed impossible by the top-sequents.

Let us start our discussion of the operational rules by looking at the NMMS left-rule for conjunction, which we repeat here for convenience:

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\text{[L}\wedge\text{]}$$

Recall that, like all of our operational rules, this rule is invertible, so that it holds not only from top to bottom but also from bottom to top. Hence, [L∧] implies that $\Gamma, A, B \succ \Delta$ is derivable if and only if $\Gamma, A \wedge B \succ \Delta$ is derivable. By the definition of truth-maker consequence, this rule is sound for truth-maker consequence just in case, in all models, if we take any state, $g \Cup d$, that is a fusion of truth-makers for each member of Γ and falsity-makers for each member of Δ, then all fusions of $g \Cup d$ with a truth-maker, a, of A and a truth-maker, b, of B are impossible if and only if all fusions of $g \Cup d$ with any truth-maker, ab, of $A \wedge B$ are impossible. Since this must hold for arbitrary states $g \Cup d$, it includes the case where $g \Cup d = \square$. Hence, it must be that $\square \Cup a \Cup b$ is impossible if and only if $\square \Cup ab$ is impossible and, so, $a \Cup b$ is impossible if and only if ab is impossible.

Now, recall the semantic clause for the truth-makers of conjunctions from above, which we repeat for convenience with the variables adjusted to match our present case.

(and+) $ab \Vdash A \wedge B$ if and only if
$\exists a, b\,(a \Vdash A$ and $b \Vdash B$ and $ab = a \Cup b)$

This clause says that the truth-makers of $A \wedge B$ are all and only the fusions of a truth-maker of A and a truth-maker of B. Thus, the states that our sequent rule requires to be co-impossible in all models are precisely those states that our semantic clause requires to be identical in all models. Because these states are identical, the results of fusing them with $g \Cup d$ are also identical. So, our semantic clause (and+) is sufficient to ensure that [L∧] is valid for truth-maker consequence.

Moreover, our semantic clause (and+) is also necessary for the validity of [L∧]. For, suppose that there were a state ab that is a truth-maker of $A \wedge B$ but is not a fusion of a truth-maker of A and a truth-maker of B. We could then find a model in which any fusion of the form $g \Cup d \Cup ab$ is impossible while fusing $g \Cup d$ with some truth-maker of A and some truth-maker of B is possible. So, in that model, $\Gamma, A, B \not\models_{\overline{TM}} \Delta$ but $\Gamma, A \wedge B \models_{\overline{TM}} \Delta$. And we can generalize this strategy to cover all models, so that we have a counterexample to [L∧].

What this means is that our semantic clause for the truth-makers of conjunctions, namely (and+), and our left-rule for conjunction in the sequent calculus, namely [L∧], have the same impact on the consequence relation. They are equivalent in their significance for the consequence relations defined in the two frameworks. Another way to look at this fact is to note that the sequents $\Gamma, A, B \succ \Delta$ and $\Gamma, A \wedge B \succ \Delta$ deem exactly the same states impossible. A corresponding result holds for all of our operational rules, namely:

Lemma 41. *For every application of an operational rule of* NMMS*, the set of states deemed impossible by the bottom-sequent is the union of the sets of states deemed impossible by the top-sequents. (Appendix, Lemma 50)*

The reason why this holds is that the states that the active sentence in the bottom-sequent contributes to the fusions that the bottom-sequent deems impossible are exactly the states that the active sentences in the top-sequents contribute to their fusions.

To see what we mean, consider the NMMS right-rule for conjunction as another example. Recall from the previous chapter that the NMMS right-rule for conjunction is the following:

$$\frac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B \qquad \Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \wedge B}\ [\mathrm{R}\wedge]$$

And recall that the semantic clause for falsity-makers of conjunctions from above is the following, repeated here for convenience:

(and−) $s \dashv\!| B \wedge C$ if and only if
$s \dashv\!| B$ or $s \dashv\!| C$ or $\exists u, t\, (u \dashv\!| B$ and $t \dashv\!| C$ and $s = u \Cup t)$

Note that the bottom-sequent, $\Gamma \succ A \wedge B, \Delta$, of our right-rule holds, by the definition of truth-maker consequence, just in case every fusion of truth-makers for Γ and falsity-makers for Δ with any falsity-maker of $A \wedge B$ is an impossible state. Now, by our semantic clauses, a falsity-maker of $A \wedge B$ is a state that is either a falsity-maker of A or a falsity-maker of B, or a fusion of such falsity-makers. But the fusions of those states with truth-makers for Γ and falsity-makers for Δ are precisely the states that are deemed impossible by the top-sequents of [R∧]. So, the set of states deemed impossible by the bottom sequent, $\Gamma \succ A \wedge B, \Delta$, is precisely the union of the states deemed impossible by the top-sequents of [R∧] according to (and−).

Similarly, for negation, since a truth-maker of A is a falsity-maker of $\neg A$, the sequents $\Gamma, A \succ \Delta$ and $\Gamma \succ \neg A, \Delta$ deem exactly the same states impossible. And since a falsity-maker of A is a truth-maker of $\neg A$, the sequents $\Gamma \succ A, \Delta$ and $\Gamma, \neg A \succ \Delta$ deem exactly the same states impossible. That corresponds to the right-rule and the left-rule for negation.

The lemma above says that this is always the case, that is, it holds not only for conjunction and negation but also for disjunction. It follows that applications of our sequent-rules are guaranteed to preserve truth-maker consequence.

Proposition 42. *All the operational rules of* NMMS *are sound for* $\models_{\overline{TM}}$, *that is, if all the top-sequents of a rule-application are TM-valid, then so is the bottom sequent. (Appendix, Proposition 51)*

Indeed, our lemma shows not only that our sequent rules are sound but also that they are complete with respect to the corresponding truth-maker clauses. To see the outlines of an argument for why this holds (see the Appendix for more details), suppose that a sequent is not derivable. Then for any rule-application that would conclude that sequent, one of the top-sequents must be underivable. And when we trace this underivability up a potential proof-tree, we will reach a sequent whose underivability depends merely on atomic sentences. It can be shown that we can construct a counterexample for such a sequent by specifying a model in which a fusion of truth-makers for its premises and falsity-makers for its conclusions is a possible state. And thanks to our lemma, the underivable sequent with which we started must deem that state in that model impossible. Hence, we have a counterexample to our underivable sequent.[33] So the sequent rules are both sound and complete with respect to the truth-maker semantics.

4.3.3 Structural Rules

This brings us to the structural rules. We won't consider permutation and contraction, but continue to think in terms of sets of sentences. Thus, we have just three structural rules: [Containment], [Weakening], and [Cut]. These are the sequent rules that correspond to (versions of) the reflexivity, monotonicity, and transitivity of consequence. They are the constraints that allow our consequence relations to encode only topologically closed reason relations. If we want to be able to theorize open reason relations, we must relax these constraints. Hence, if we want to formulate versions of truth-maker theory that can capture open reason relations, it is crucial for us to understand what corresponds to these structural constraints in truth-maker theory and how we can relax them. We repeat the structural sequent rules here for convenience:

$$\frac{}{\Gamma, A \succ A, \Delta}\ [\text{Containment}]$$

$$\frac{\Gamma \succ \Delta}{\Gamma, \Theta \succ \Delta, \Lambda}\ [\text{Weakening}] \qquad \frac{\Gamma \succ \Delta, A \qquad A, \Gamma \succ \Delta}{\Gamma \succ \Delta}\ [\text{Cut}]$$

As already intimated, the three structural rules correspond exactly to the three constraints on possible states. Exclusivity corresponds to [Containment]. Downward-Closure corresponds to [Weakening]. And Exhaustivity corresponds to [Cut]. Let us run through the arguments for the three rules.

Recall that Exclusivity says that if $s \in |p|^+$ and $t \in |p|^-$, then $\forall u(s \Cup t \Cup u \notin S^\diamond)$, that is, if you fuse a truth-maker and a falsity-maker of an atomic sentence with any other state, then the result is always an impossible state. If that holds in a model, then clearly $\Gamma, p \models_{TM} p, \Delta$ in that model. And if it holds in all models, then $\Gamma, p \models_{LTM} p, \Delta$. But that is precisely what the [Containment] rule allows us to prove; for it allows us to derive $\Gamma, p \succ p, \Delta$. Moreover, for the other direction, in order to ensure that all the sequents of the form $\Gamma, p \succ p, \Delta$ hold according to the logic of truth-maker consequence, we must accept Exclusivity as a constraint on possible states.

Downward-Closure says that if $s \in S^\diamond$ and $t \sqsubseteq s$, then $t \in S^\diamond$, that is, if a state is possible, then all of its parts are possible. So, if a state is impossible, then all the states of which it is a part are impossible. A sequent $\Gamma \succ \Delta$ is TM-valid, in a model, just in case all the states that are fusions of truth-makers of Γ and falsity-makers of Δ are impossible. If that is the case and Downward-Closure holds, then all the states that include such a fusion as a part are also impossible. Now, such a fusion is part of every state that is deemed impossible by a sequent of the form $\Theta, \Gamma \succ \Delta, \Sigma$. For these states are fusions of truth-makers of Γ and falsity-makers of Δ with some further states, namely truth-makers of Θ and falsity-makers of Σ. Hence, if $\Gamma \succ \Delta$ is TM-valid, then so is $\Theta, \Gamma \succ \Delta, \Sigma$. Thus, Downward-Closure is sufficient for the soundness of [Weakening].

Downward-Closure is also necessary for the soundness of [Weakening] if we assume the expressibility of propositions. For suppose that Downward-Closure fails, so that there is an impossible state, s, that is part of a possible state, $s \Cup t$. If we can find a sequent, $S\succ$, that deems only the impossible state s, impossible, and also a sentence, T, whose only truth-maker is t, then $S \models_{TM}$ but $S, T \not\models_{TM}$. And that gives us a counterexample to [Weakening]. Hence, Downward-Closure and [Weakening] are equivalent regarding their effects on the consequence relation in their respective frameworks.

Lastly, remember that Exhaustivity says that if a state, u, is possible, then either $\exists s \in |p|^+ \ (u \Cup s \in S^\diamond)$ or $\exists t \in |p|^- \ (u \Cup t \in S^\diamond)$. That is, for any atomic sentence, a possible state can always be extended to another possible state by fusing it either with a truth-maker or with a falsity-maker of the atomic sentence. To see that this corresponds to [Cut], it is helpful to read [Cut] contrapositively: if $\Gamma \not\succ \Delta$, then either $\Gamma \not\succ p, \Delta$ or $\Gamma, p \not\succ \Delta$. If we think of this in terms of TM-validity, it says that if there is a possible fusion of truth-makers of Γ and falsity-makers of Δ, then there is either a possible fusion of truth-makers of $\Gamma \cup \{p\}$ and falsity-makers of Δ or there is a possible fusion of truth-makers of Γ and falsity-makers of $\Delta \cup \{p\}$. Exhaustivity ensures that this holds; so it ensures that [Cut] is sound.

Moreover, assuming the expressibility of propositions again, Exhaustivity is not only sufficient but also necessary for the soundness of [Cut]. For suppose that there is a possible state, u, such that neither $\exists s \in |p|^+ \ (u \Cup s \in S^\diamond)$ nor $\exists t \in |p|^- \ (u \Cup t \in S^\diamond)$. If u is the only truth-maker of U and we let u be possible, then $U \not\models_{\overline{TM}}$ but $U, p \models_{\overline{TM}}$ and $U \models_{\overline{TM}} p$. And that is a counterexample to [Cut]. Hence, Exhaustivity and [Cut] are equivalent with respect to their effect on consequence.

So, not only do the operational rules of our sequent calculus correspond exactly to the semantic clauses in truth-maker theory but the structural rules correspond exactly to constraints on possible states. We can summarize our results regarding the correspondence between structural rules and constraints on possible states in the following proposition.

Proposition 43. *The rules [Containment], [Weakening], and [Cut] preserve TM-validity if and only if possible states obey Exclusivity, Downward-Closure, and Exhaustivity respectively. (Appendix, Proposition 53)*

Putting these results together suffices to show that $\models_{\overline{LTM}}$ coincides with classical propositional consequence, because the sequent calculus that includes all of our operational and structural rules is a formulation of classical propositional logic, $\models_{\overline{CL}}$, and it is also sound and complete with respect to $\models_{\overline{LTM}}$.

Proposition 44. *If we impose Exclusivity, Exhaustivity, and Downward-Closure, then $\Gamma \models_{\overline{LTM}} \Delta$ if and only if $\Gamma \models_{\overline{CL}} \Delta$. (Appendix, Proposition 54)*

Of course, our goal here is not to recover classical logic in truth-maker theory. Rather, the interest of this result lies in the fact that we have obtained it by finding a correspondence between the parts of our truth-maker theory and our sequent calculus. We now have a clearer picture of how truth-maker theory and the sequent calculus rules from the previous chapter correspond to each other. And we can now turn to the question how we can codify open reason relations in truth-maker theory.

4.4 Accommodating Open Reason Relations

We have seen in the previous section that the structural rules of [Cut] and [Weakening] correspond to constraints on possible states that are familiar from Fine, namely Exhaustivity and Downward-Closure.[34] This suggests that if we want to allow for reason relations that are open in the sense that they are not constrained by Transitivity or Monotonicity, we can do that in truth-maker theory by dropping Exhaustivity or Downward Closure. The aim of this section is to show that this is indeed the case.

4.4.1 Nontransitive Truth-Maker Consequence

Let us then formulate a nontransitive version of truth-maker consequence by dropping Exhaustivity. If we consider logical truth-maker consequence, then this change by itself won't have any effect on our logical consequence relation because [Cut] is admissible. In terms of truth-maker theory: Exhaustivity is redundant if Exclusivity is the only constraint on possible states that holds in all models. There is, however, a well-known way to make [Cut] fail in a useful and interesting way, namely the so-called strict/tolerant logic STT, which includes a transparent truth-predicate and was developed as a response to the semantic paradoxes (Cobreros et al., 2013, 2012; Ripley, 2012). Hence, recovering STT in truth-maker theory not only gives us a new kind of semantics for nontransitive logics but is also an interesting test-case for our idea that we can allow for nontransitive reason relations in truth-maker theory by dropping Exhaustivity.[35]

In the absence of any additional notions like truth, the logic ST is equivalent to the NMMS extension of a base that consists of all and only the instances of Containment for a given atomic language, which we have seen to coincide with classical logic in the previous chapter. This logic can be extended to STT by adding a truth predicate, as follows: Let's add to our object language a canonical name $\bar{A}$ for every sentence A and a truth-predicate, *Tr*, for which we add the following sequent rules to NMMS.[36]

$$\frac{\Gamma, A \succ \Delta}{\Gamma, Tr(\bar{A}) \succ \Delta}\,[\mathrm{Lt}] \qquad\qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ Tr(\bar{A}), \Delta}\,[\mathrm{Rt}]$$

Definition 45 (STT-calculus)**.** The STT-calculus is the sequent calculus that is like NMMS, over a base that consists of all and only the instances of Containment for the atomic sentences, except that it includes the rules [Lt] and [Rt]. We say that $\Gamma \mathrel{|\!\!\!\underline{}_{\mathsf{STT}}} \Delta$ if and only if the sequent $\Gamma \succ \Delta$ is derivable in this calculus.[37]

Let us also allow for self-reference by allowing sentences that include their own names. Thus, we can formulate a Liar sentence, $\neg Tr(\bar{\lambda})$, whose name is $\bar{\lambda}$. This sentence says of itself that it is not true. Note that λ is everywhere intersubstitutable with $\neg Tr(\bar{\lambda})$ *salva consequentia*.[38] Since $Tr(\bar{\lambda})$ is an atomic sentence, [Containment] yields $Tr(\bar{\lambda}) \succ Tr(\bar{\lambda})$. Using

[Lt], [Rt], our negation rules, and the intersubstitutability of $\neg Tr(\bar{\lambda})$ and λ, we can derive $\succ\lambda$ and $\lambda\succ$ (Ripley, 2013). Applying [Cut] now yields the empty sequent. However, [Cut] is not a rule of STT. Hence, we can accept that $\succ\lambda$ and $\lambda\succ$ are derivable without this rendering our consequence relation trivial. This is the essence of the nontransitive approach to the Liar Paradox.

If the isomorphism that we explained above lives up to our promises above, we can now make the analogous changes in our truth-maker theory. And, indeed, we can do this (see Hlobil, 2022a).[39] Namely, we can do this by adding a truth predicate to our truth-maker theory, which is governed by the following semantic clauses:

(tr+) $s \Vdash Tr(\bar{A})$ if and only if $s \Vdash A$

(tr−) $s \dashv\!\!\| \, Tr(\bar{A})$ if and only if $s \dashv\!\!\| \, A$

We allow for self-reference as in STT, so that the language of our truth-maker theory includes a liar sentence $\neg Tr(\bar{\lambda})$, whose name is $\bar{\lambda}$. If our truth-maker theory includes Exhaustivity, this implies that all states in all models are impossible. For, suppose that $s \Vdash \lambda$, that is, $s \Vdash \neg Tr(\bar{\lambda})$. Then $s \dashv\!\!\| \, Tr(\bar{\lambda})$ and, hence, $s \dashv\!\!\| \, \lambda$. So, a state makes the liar sentence true if and only if it makes the liar sentence false. Thus, by Exclusivity, any state that includes a truth-maker or a falsity-maker of the liar sentence is impossible. But if we can extend every possible state to a possible state by either adding a truth-maker or a falsity-maker of the liar sentence but there is no possible state that includes a truth-maker or a falsity-maker of the liar sentence, then there cannot be any possible state. Given the definition of truth-maker consequence, it follows that every set of premises implies every set of conclusions. The solution to this paradox that mirrors the solution of STT is, of course, to reject Exhaustivity. Let us define this version of truth-maker theory explicitly.

Definition 46 (TM-consequence for base STT). Let $\models^{STT}_{LTM}$ be the consequence relation just like $\models_{LTM}$ except that we include models that violate Exhaustivity and the language has a truth-predicate whose interpretation obeys the clauses (tr+) and (tr−).

These changes have exactly the effects that we have envisaged. The semantic clauses have the same effect on consequence as the sequent rules for the truth-predicate, and dropping [Cut] and Exhaustivity have also the same effect. Hence, this version of (logical) truth-maker consequence coincides with the consequence relation of STT.

Proposition 47. $\Gamma \mid\!\!\frac{}{\mathsf{STT}} \Delta$ *if and only if* $\Gamma \models^{STT}_{LTM} \Delta$. *(Appendix, Proposition 59)*

Notice that the two formal systems don't only match in their consequence relations. Rather, they match in a piece-by-piece fashion, in the way explained in the previous subsection. Indeed, all we did was to use the correspondence from the previous subsection to formulate a version of truth-maker consequence that mirrors the sequent calculus for STT.[40]

We can illustrate this by spelling out how the philosophical interpretation of failures of [Cut] can be translated from normative bilateralism into truth-maker theory. For normative bilateralism, the intuitive idea behind the rejections of [Cut] is that any position that includes an assertion or a denial of the liar sentence is out-of-bounds. When we translate this into truth-maker bilateralism, the result is this: Any state that includes a verifier or a falsifier of the liar sentence is impossible. Given any possible state, this yields a violation of Exhaustivity. Since the world is a possible state, we can express the idea by saying that the world cannot contain anything that makes the liar sentence either true or false. Just as we should neither assert nor deny the liar sentence, so the world can neither verify nor falsify it.

We have thus formulated a truth-maker semantics for strict/tolerant logic, and this illustrates how we can allow for nontransitive reason relations in truth-maker theory. Now, the reason why we rejected transitivity in the previous chapters had nothing to do with the semantic paradoxes. We argued rather that once we free ourselves from the constraint of Monotonicity on reason relations, our language will have crippling expressive limitations unless we also allow for nontransitive reason relations. Hence, we must consider whether the strategy for allowing for open reason relations in truth-maker theory can yield the desired results when we move away from STT as a test-case.

4.4.2 Nonmonotonic Truth-Maker Consequence

We now want to make the final step to show that the framework of truth-maker consequence is isomorphic to the theory that we presented in the previous chapter. We want to show that if we reject Monotonicity and Transitivity in our truth-maker theory and we can add material implications among atomic sentences, then the result is the same as the logical extension of these material implications by the rules of NMMS from the previous chapter.

As we have seen in the previous chapter, NMMS allows for failures of Monotonicity and Transitivity. We have seen in the previous section that Monotonicity corresponds to Downward-Closure in truth-maker theory, and Transitivity corresponds to Exhaustivity. Hence, in order to mirror NMMS within truth-maker theory, we drop Downward-Closure and Exhaustivity. Moreover, we must add material, non-logical implications among atomic sentences to our truth-maker theory in order to mirror the use of base vocabularies in NMMS.

Recall that we started in the previous chapter with a material base vocabulary, which consists of an atomic language $\mathfrak{L}_{\mathfrak{B}}$ and a consequence relation $\mid\!\sim_{\mathfrak{B}}$ over subsets of that atomic language. The base consequence relation encodes non-logical reason relations of both fundamental varieties: reasons-for and reasons-against. Let us quickly rehearse some examples of the ideas from the previous chapter. Let's suppose that $c =$ "This is a chair" implies $s =$ "You can sit on this." Hence, $c \mid\!\sim_{\mathfrak{B}} s$. Since this implication is defeated if we add the additional premise $b =$ "This is broken," the corresponding pair is not in the material base. That is, $c, b \not\mid\!\sim_{\mathfrak{B}} s$. Similarly, since "This is a chair" is incompatible with $v =$ "This is a violin," we have $c, v \mid\!\sim_{\mathfrak{B}}$. But since this incoherence is cured by adding $a =$ "This is part of an art project that makes pieces of furniture that are musical instruments," we say $c, v, a \not\mid\!\sim_{\mathfrak{B}}$. Thus, our two reason relations, implication and incompatibility, are both defeasible, and we can model this in our formalism with an appropriate material base consequence relation. This is all familiar from the previous chapters. Moreover, we showed in the previous chapter how we can add logical vocabulary to such a material base, namely by closing the material base under the operational rules of NMMS. We saw in the previous chapter that if we restrict our material bases to those that obey Containment, all consequence relations defined by such a calculus include all of classical logic. And we showed how to add various kinds of object language operators to make explicit, in the object language, various local features of such consequence relations.

Our question now is how we can find truth-maker formulations of these consequence relations. The first step is to ensure that a truth-maker consequence relation includes all the atomic sequents in a given material base $\mathfrak{B}$. We can do that by stipulating, for every pair $\langle \Gamma, \Delta \rangle \in \mid\!\sim_{\mathfrak{B}}$, that any fusion of verifiers for everything in Γ and falsifiers for everything in Δ is impossible. The following definition does that.

Definition 48 (TM-consequence for base $\mathfrak{B}$). $\models^{\mathfrak{B}}_{TM}$ is the consequence relation that is just like $\models_{TM}$ except that the only constraint on possible states is that $s \notin S^{\diamond}$ if and only if $s = \biguplus\{g_1, ..., g_n, d_1, ..., d_m\}$ such that $\langle \Gamma, \Delta \rangle \in \mid\!\sim_{\mathfrak{B}}$ and $\Gamma = \{\gamma_1, ..., \gamma_n\}$ and $\Delta = \{\delta_1, ..., \delta_m\}$ and $\forall 1 \leq i \leq n\, (g_i \in |\gamma_i|^+)$ and $\forall 1 \leq i \leq m\, (d_i \in |\delta_i|^-)$.

This definition says that, for every sequent in the material base, every state that the sequent deems impossible is impossible in our model. We could quantify over all truth-maker models that meet this constraint, but we don't really need to do that in order to define $\models^{\mathfrak{B}}_{TM}$. The model in which all and only those states are impossible that must be impossible for the material base to hold already includes all the counterexamples to all invalid sequents.

If we understand consequence in this way, then the failures of Monotonicity in the examples above look as follows: In the absence of any defeating facts, the state of something being a chair and the state of one being unable to sit on it are incompatible, that is, their fusion is impossible. However, if we add to this fusion the state that the object in question is broken, then the states are no longer incompatible, that is, the fusion of all three states is a possible state. Similarly, the states of something being a chair and the state of it being a violin are incompatible, that is, their fusion is an impossible state. However, if we fuse this state with the state of the object being part of an art project that makes pieces of furniture that are musical instruments, then the resulting state isn't impossible. The thought is that certain states can make states compatible with each other that are otherwise incompatible. These states need, as it were, the further state in order to fit together.

We know that the semantic clauses in truth-maker theory correspond to the operational rules in sequent calculi. So adding logical vocabulary that obeys the semantic clauses has the same effect on consequence as closing the base sequents under the corresponding sequent rules. And since we didn't enforce Exhaustivity and Downward-Closure but insisted on Exclusivity, the resulting consequence relation is guaranteed to obey Containment, but it may be nontransitive and nonmonotonic. In fact, the result is exactly what you might expect: the sequent calculus version of our current logic and the truth-maker consequence relation of it coincide if the material bases are the same.

Proposition 49. $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$ *just in case* $\Gamma \models^{\mathfrak{B}}_{TM} \Delta$. *(Appendix, Proposition 60)*

So we now have a way to construct a truth-maker semantics for any nonmonotonic logic that can be obtained by closing an arbitrary set of atomic sequents under given operational rules. Thus, we can formulate our account of consequence relation in terms of the rules of NMMS from the previous chapter within truth-maker theory in a surprisingly straightforward way. We simply take the semantic clauses that correspond to the desired sequent rules, and we stipulate the material base by way of a constraint on possible states. If we restrict ourselves to bases that obey Containment, then the result will include all classically valid sequents. If we want to enforce a structural rule, we add the corresponding constraint on possible states. If we want to add another bit of logical vocabulary, we read off the semantic clauses from the sequent rules. The clause for truth-makers is given by the left rule, and the clause for falsity-makers is given by the right rule. We saw an example of this when we looked at the clauses for the truth-predicate.

The isomorphism between the theories ensures that parallel tweaks yield parallel results. In this way, we can see which semantic-representationalist rules and principles correspond to the rules and principles of a particular pragmatic-normative account of content and reason relations, and *vice versa*. And what connects the two sides of this correspondence to each other is the notion of representation. For, a particular pragmatic-normative account corresponds to a particular semantic-representationalist account just in case the rational form attributed to a sentence by the pragmatic-normative account is the same as the rational form attributed to the worldly proposition expressed by that sentence according to the semantic-representationalist account.

The upshot of all this is that the sentences that occur in discursive acts that are governed by the norms that are codified in a consequence relations defined by $\mathsf{NMMS}_{\mathfrak{B}}$ share their rational form with the worldly propositions that these same sentences express according to the truth-maker model that defines the corresponding relation $\overset{\mathfrak{B}}{\underset{TM}{\models}}$. That is, the modal relations of exclusion among assertions and denials of sentences and among their truth-makers and falsity-makers is isomorphic, supposing that we use the same base relation in $\mathsf{NMMS}_{\mathfrak{B}}$ and in truth-maker theory. Thus, we can now see how a pragmatic-normative account of content and reason relations can be isomorphic to a semantic-representationalist account of content and reason relations. And we can think of the relation of representation as holding precisely if and when such an isomorphism indeed holds in a particular case. Moreover, we have seen how we can ensure that such an isomorphism holds for logically complex sentences, supposing that it holds for atomic sentences. All this can be done no matter how radically open-structured (substructural) the reason relations that articulate the base vocabulary are.

4.5 Conclusion

We began the Introduction to this book by pointing out that two principal traditions in the philosophy of language are distinguished by whether they focus to begin with on the *use* of linguistic expressions, or on their *meanings*. These topics are addressed by two different disciplines: pragmatics (in a broad sense) and semantics. We urged thinking about these in terms of the metavocabularies they use to discuss the aspects of the base vocabularies they target. In Chapter One, we presented our version of a bilateral normative pragmatic metavocabulary. It specifies norms governing discursive practices of using declarative sentences in speech acts of asserting and denying, which manifest doxastic attitudes of acceptance and rejection. In the present chapter, we discussed the sophisticated contemporary modal truth-maker formal semantic metavocabulary. It appeals both to matter-of-factual worldly states, divided into the possible and the impossible, and to their mereological relations to one another, to determine what makes declarative sentences true or false, depending on how things objectively are. Though the pragmatic and semantic theoretical enterprises can be, and have been, conducted each in their own terms, substantially independently of each other, it is also important to understand the relations between the activities of talking and thinking (using expressions as representings of something) and what is being talked or thought about (what is thereby represented). Thought of in these terms, understanding the relations between what is expressed by pragmatic metavocabularies and what is expressed by semantic metavocabularies is essential to understanding the intentional nexus between (attempted) knowings and what is known.

We have adopted an indirect approach to this challenge. We regard (the right kind of) pragmatic and semantic metavocabularies as providing two different conceptual and explanatory perspectives on a common object: reason relations of implication and incompatibility. They should be understood as being metavocabularies specifying the reason relations of some autonomous base vocabulary. They can therefore be thought of as metavocabularies of reason, or rational metavocabularies, in the sense in which we use that term. On the pragmatic side, the two kinds of reason relation determine what claims should be accorded the practical significance of being reasons for and against what others (in the broad sense of "reasons" operative in this book). They articulate (aspects of) standards for assessing the success of speech acts of justifying, seeking rationally to secure entitlements to constellations of commitments to accept or reject, and of speech acts of challenging, rationally contesting such entitlements. On the semantic side, the two kinds of reason relation show up as modal relations determining how worldly states are possible or impossible relative to one another.

Of course each sort of metavocabulary also talks about things that are not reason relations: about speech acts and their relations to two kinds of normative status, in the one case, and about states and their fusions and their absolute and relative possibility and impossibility, in the other case. It is that further conceptual material (extrinsic to the base or object-language vocabulary for which they are metavocabularies) that pragmatic and semantic theories in these metavocabularies deploy to *explain* relations of implication and incompatibility, each in its own way. Each offers an account of what relations of rational implication and incompatibility are, and of what determines what implies and is incompatible with what. The pragmatic and semantic explanations of reason relations differ not only in the conceptual raw materials they appeal to, but also in the kind of modal relations that are essential to the distinct sorts of account they articulate. The fundamental modal vocabulary employed by the pragmatics consists of normative terms such as "commitment" and "entitlement." The fundamental modal vocabulary employed to specify the metaphysical universe in terms of which semantic interpretants are defined consists of alethic terms such as "possible" and "impossible."

In this chapter we showed that in spite of these substantial differences in explanatory raw materials and form there is a structural correspondence between the pragmatics-first approach as we pursued it in the previous two chapters, and a semantics-first approach as it is pursued in truth-maker theory. Furthermore, the correspondence is a conceptually and technically fruitful one. On the pragmatic side, we fill in the discursive social practices that confer content on the notion of "out-of-boundness," and further articulate the distinction and relations between the two kinds of normative status implicit in bilateralism: the commitments that make up a position and the notion of normative impossibility or "out-of-boundness" understood as preclusion of entitlement to such a constellation of commitments. By offering a truth-maker semantics for the expressively LX logic NMMS, we show how the truth-maker framework can be extended in a philosophically motivatable and formally tractable way to codify radically substructural nonmonotonic and nontransitive reason relations of implication and incompatibility. When so used, the apparatus tells us exactly what the metaphysical structure of worldly states must be like in order properly to be thought about using nonmonotonic and nontransitive reasoning practices.

In fact, what is on offer is more than just a correspondence. We have shown how to define a detailed isomorphism relating the pragmatic and semantic metavocabularies on which we have focused. They are isomorphic precisely at the level of reason relations. That isomorphism defines higher-order representational norms governing the relations between the pragmatic use of declarative sentences and their semantic meanings. Very roughly, in order to represent worldly propositions with sentences, the conceptual norms governing discursive practice must be such that according to them, the pragmatic-normative structure *ought* to match the alethic-modal structure. More specifically, properly to represent relations of worldly consequence, language users should in practice take it that they cannot be entitled to accept all the sentences in a set Γ and reject all the sentences in Δ just in case the fusion of any set of truth-makers of all of Γ with any set of false-makers of all of Δ is always an impossible state. (On the side of incompatibility, sets of commitments to accept to which one cannot be jointly entitled should line up with sets of sentences such that every fusion of truth-makers for each of them is an impossible state.)

We have been using the term "rational form" to talk about the roles sentences and what the sentences express play with respect to reason relations. Insofar as the higher-order representational norm defined by that isomorphism holds across the board, the rational form conferred on sentences by their use as specified in the normative bilateral metavocabulary and the rational form conferred on sentences by their worldly truth-makers and false-makers coincide. That is what makes it possible for declarative sentences to say of what is, that it is, and of what is not, that it is not. That discursive acts and worldly states can share their rational forms allows us to see that our discursive acts (representings) and the worldly states that they are about (represent) can have something in common. And what they have in common is that in virtue of which our discursive acts have the contents they have and in virtue of which the worldly states are the kinds of states they are. The rational pragmatic-semantic isomorphism accordingly fills in important details of what in the Introduction we called "bimodal conceptual realism."

The vantage point afforded by defining this isomorphism shows how to understand as complementary rather than competing accounts the very different explanations pragmatic and semantic metavocabularies offer of what it is for sets of sentences to imply or be incompatible with other sets of sentences. For it permits the precise specification of the topic common to them, the object on which they offer complementary perspectives, what they turn out both to present, albeit in different guises—namely reason relations and the rational forms they functionally define. Isomorphisms define equivalence relations that support the introduction by abstraction of new terms corresponding to equivalence classes of expressions in the vocabularies from which they are abstracted. Though we found our way to the conception of reason relations via the different specifications of it in (the right kind of) pragmatic and semantic metavocabularies, we are now in a position to move beyond those perspectives on reason relations. Reason relations of implication and incompatibility, and the rational forms they define and articulate, are introduced by abstraction, picked out by equivalence classes with respect to the pragmatic-semantic isomorphism. They are what has been our topic all along. The constructions and demonstrations of this chapter have advanced the story by putting us in a position properly to delineate the objects these different metavocabularies provide perspectives on, in a way that is unalloyed by admixture with extraneous pragmatic or semantic concepts, important and informative as those otherwise are.

The abstract notion of rational form brought into view by the isomorphism between consequence relations specified in deontic normative pragmatic and alethic modal semantic metavocabularies suggests that we should investigate those abstract common forms in their own right. For if what we have said so far is correct, this common form is the structure of reason relations in general—not only as they appear in our discursive practice of giving and asking for reasons but also as they articulate the worldly relations about states to which our conceptual norms correspond. Accordingly, in the next chapter we introduce a metavocabulary that is an intrinsic metavocabulary specifying rational forms understood as roles with respect to reason relations considered abstractly: conceptual propositional roles. We provide an implication-space semantics for conceptual roles. It is a way of working out the aspirations of semantic inferentialism, as the construction of NMMS as universally LX (extending even to arbitrary open-structured base vocabularies) in Chapter Three is a way of working out the aspirations of logical expressivism.

In calling the implication-space model-theoretic semantics for reason relations and conceptual roles an "intrinsic" metavocabulary, we are contrasting it with both pragmatic and semantic metavocabularies. They count as "extrinsic" metavocabularies because their explanations of reason relations appeal to conceptual raw materials that are not drawn entirely from the base vocabularies whose reason relations are the targets of their explanations. In Chapter Three we considered logical vocabulary as being elaborated entirely from those base vocabularies, importing no extrinsic material to them. In using the base vocabularies practitioners already know how to do everything they need to know how to do to deploy the logical vocabulary. Logical vocabulary is in this sense an intrinsic metavocabulary for expressing the reason relations of base vocabularies it extends. We saw that its distinctive function is expressive and explicative rather than explanatory. Logical vocabulary serves to make explicit the reason relations of the base vocabulary, in the sense of saying what follows from what and what is incompatible with what, rather than aiming at explaining why or in what sense things stand in the reason relations they do—for instance, by appeal to features of discursive practice, or of worldly states. The implication-space semantics characterizes rational forms (propositional contents) directly in terms of roles with respect to reason relations. It, too, is an intrinsic-explicative rational metavocabulary, rather than an extrinsic-explanatory one. It completes the set of four rational metavocabularies we employ in this book as tools for specifying, investigating, and manipulating reason relations and conceptual roles.

Our discussion in the next chapter of the implication-space semantics for conceptual roles also adds two further connections among those four kinds of rational metavocabulary. In this chapter we specified two such relations precisely, in the form of the isomorphism between pragmatic and semantic metavocabularies, and the construction of a truth-maker semantics that is sound and complete for the logical vocabulary of NMMS. Chapter Five shows how to define a more abstract implication-space version of any truth-maker semantic metavocabulary, and also shows that it, too, is sound and complete for the logic NMMS, even for radically open-structured nonmonotonic and hypernonmonotonic, nontransitive base vocabularies. The chapter closes by building on the implication-space semantic vocabulary a still more abstract characterization of conceptual roles, including partial conceptual roles such as premissory and conclusory roles. There we show how to use that metavocabulary to draw connections between this abstract form of reason relations and a number of familiar and interesting logics, including Linear Logic, Priest's Logic of Paradox, and Strong Kleene Logic.

The four metavocabularies for specifying reason relations and conceptual roles or rational forms are divided into the two extrinsic-explanatory rational metavocabularies, bilateral-pragmatic and truth-maker-semantic, and the two intrinsic-explicative rational metavocabularies, logical and implication-space. Along with the criss-crossing relations among them that we demonstrate and explore, this is the structure that we propose for a functionalist specification of reason relations and rational forms, in terms of the role those phenomena play as the objects of all these intricately interconnected kinds of metavocabulary. Proceeding in this way is pursuing the strategy that above we called "meta-rational functionalism": defining the object of study by the reason relations that articulate the meta-meta-vocabulary (metavocabulary for rational metavocabularies) that is the use-language of this book. We turn now to consideration of the abstract implication-space semantics for conceptual roles that is the fourth vertex of the tetrahedral structure of rational metavocabularies.

4.6 Appendix

Lemma 50. *For every application of an operational rule of* NMMS, *the set of states deemed impossible by the bottom-sequent is the union of the states deemed impossible by the top-sequents.*

Proof. We do the case for conjunction; the proofs for negation and disjunction are analogous. For [L$\wedge$]: Note that by our semantic clauses $|A \wedge B|^+ = \{s : \exists a \in |A|^+ \exists b \in |B|^+ (s = a \uplus b)\}$. Hence, for any Γ and Δ, we have $\{g \uplus d \uplus a \uplus b : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $a \in |A|^+$ and $b \in |B|^+\} = \{g \uplus d \uplus s : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $s \in |A \wedge B|^+\}$. So the states deemed impossible by $\Gamma, A, B \succ \Delta$ are identical to those deemed impossible by $\Gamma, A \wedge B \succ \Delta$.

Similarly for [R$\wedge$], note that $|A \wedge B|^- = |A|^- \cup |B|^- \cup \{s : \exists a \in |A|^- \exists b \in |B|^- (s = a \uplus b)\}$. Therefore, $\{g \uplus d \uplus s : g \in |\bigwedge \Gamma|^+$ and

$d \in |\bigvee \Delta|^-$ and $s \in |A \wedge B|^-\} = \{g \Cup d \Cup a : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $a \in |A|^-\} \cup \{g \Cup d \Cup b : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $b \in |B|^-\}$ $\cup \{g \Cup d \Cup c : g \in |\bigwedge \Gamma|^+$ and $d \in |\bigvee \Delta|^-$ and $c \in \{x : \exists a \in |A|^- \exists b \in |B|^- (x = a \Cup b)\}\}$. Hence, the states deemed impossible by $\Gamma \succ \Delta, A \wedge B$ are the union of those deemed impossible by $\Gamma \succ \Delta, A$ and $\Gamma \succ \Delta, B$ and $\Gamma \succ \Delta, A, B$. ■

Proposition 51. *All operational rules of* NMMS *are sound for* $\models_{TM}$, *that is, if all the top-sequents are TM-valid, then so is the bottom sequent.*

Proof. Immediate from Lemma 50. ■

Proposition 52. *If a model is a counterexample to the top-sequent of an application of an operational rule of* NMMS, *then the model is also a counterexample to the bottom-sequent.*

Proof. By Lemma 50, if a state deemed impossible by a top-sequent is possible in $\mathcal{M}$, then a state deemed impossible by the bottom sequent is possible in $\mathcal{M}$. ■

Proposition 53. *The rules [Containment], [Weakening], and [Cut] are sound for* $\models_{TM}$ *if and only if possible states obey Exclusivity, Downward-Closure, and Exhaustivity respectively.*

Proof. Downward-Closure and [Weakening]: Suppose that $\Gamma \models_{TM} \Delta$. Then any fusion of verifiers of everything in Γ and falsifiers of everything in Δ is impossible. By Downward-Closure, all states that include any such fusion as a part are also impossible. Hence, $\Theta, \Gamma \models_{TM} \Delta, \Sigma$. For the other direction, suppose that [Weakening] is sound and that $s \in S^\diamond$ and $t \sqsubseteq s$. In accordance with Assumption 39, let the sentence S be such that s is the unique truth-maker of S, and let the sentence T be such that t is the unique truth-maker of T. Since $s \in S^\diamond$ and $t \sqsubseteq s$, we know that $s \Cup t = s$ and $S, T \not\models_{TM}$. By the contrapositive of [Weakening], $T \not\models_{TM}$. Hence, $t \in S^\diamond$.

Exclusivity and [Containment]: Let $s \in |p|^+$ and $t \in |p|^-$, then Exclusivity implies $\Gamma, p \models_{TM} p, \Delta$. So [Containment] is valid. For the other direction, by the validity of [Containment], for any Γ and Δ, we have $\Gamma, p \models_{TM} p, \Delta$. So $\forall u(s \Cup t \Cup u \notin S^\diamond)$.

Exhaustivity and [Cut]: Suppose that $\Gamma \not\models_{TM} \Delta$ and let u be a state witnessing this fact, i.e., a state that is a fusion of verifiers of everything in Γ and falsifiers of everything in Δ such that $u \in S^\diamond$. By Exhaustivity, $\exists s \in |p|^+ (u \Cup s \in S^\diamond)$ or $\exists t \in |p|^- (u \Cup t \in S^\diamond)$. Therefore, either $p, \Gamma \not\models_{TM} \Delta$ or $\Gamma \not\models_{TM} \Delta, p$. But that is just what is required for the contrapositive of [Cut]. For the other direction, suppose that [Cut] is valid for $\models_{TM}$. Let u be possible; and, in accordance with Assumption 39, let Γ be a set such that u is the unique state that is a fusion of verifiers for everything in Γ. Hence, $\Gamma \not\models_{TM}$. By the validity of [Cut], either $p, \Gamma \not\models_{TM}$ or $\Gamma \not\models_{TM} p$. Hence, either $\exists s \in |p|^+ (u \Cup s \in S^\diamond)$ or $\exists t \in |p|^- (u \Cup t \in S^\diamond)$. ■

Proposition 54. *If we impose Exclusivity, Exhaustivity, and Downward-Closure, then* $\Gamma \models_{LTM} \Delta$ *if and only if* $\Gamma \models_{CL} \Delta$.

Proof. It suffices to show that NMMS over a base that consists of all and only the atomic instances of Containment is sound and complete with respect to both consequence relations, $\models_{LTM}$ and $\models_{CL}$. We have seen in the previous chapter that NMMS over a base that consists of all and only the atomic instances of Containment is sound and complete with respect to $\models_{CL}$. For $\models_{LTM}$, we know that soundness holds from Propositions 53 and 51. For completeness, suppose that there is no proof of $\Gamma \succ \Delta$. Hence, a proof-search for $\Gamma \succ \Delta$ yields an atomic sequent, $\Gamma_0 \succ \Delta_0$, where $\Gamma_0 \cap \Delta_0 = \emptyset$. Let $\mathcal{M}$ be a model in which $s \in S^\diamond$ and $s = u \sqcup t$ such that $u \Vdash \Gamma$ and $t \dashv\!\mid \Delta$. This is a counterexample to $\Gamma_0 \succ \Delta_0$. By Proposition 52, it follows that it is also a counterexample to $\Gamma \succ \Delta$. Since $\Gamma_0 \cap \Delta_0 = \emptyset$, such a model isn't ruled out by Exclusivity, which is the only principle that could rule out such a model. So, $\mathcal{M}$ shows that $\Gamma \not\models_{TM} \Delta$. ■

The relation between STT and the truth-maker theory in this chapter is investigated in (Hlobil, 2022a). Here we merely sketch the proof-ideas. In order to prove the completeness of the STT calculus with respect to $\models^{STT}_{LTM}$, we cannot use the technique of proof-searches because proof-searches are no longer guaranteed to terminate. Hence, we follow Ripley (2013, 162–63) in using the technique of (possibly infinite) reduction trees from Takeuti (1987).

Definition 55 (Reduction tree). The reduction tree for a sequent, $\Gamma \succ \Delta$, is the possibly infinite tree that results from starting with $\Gamma \succ \Delta$ as the root of the tree and then extending at each stage each top-most sequent of the tree as follows, until all branches are closed or else extending the tree ω-many times: (i) If the sequent is an axiom, i.e., is such that the left and the right side share an atomic sentence, then the branch remains unchanged and is closed. (ii) If the sequent has the form $\Gamma, \neg A \succ \Delta$ or $\Gamma \succ \neg A, \Delta$ and no reduction has been applied to $\neg A$ in previous stages, they reduce to $\Gamma, \neg A \succ A, \Delta$ and $\Gamma, A \succ \neg A, \Delta$, respectively. (iii) If the sequent has the form $\Gamma, A \wedge B \succ \Delta$ and no reduction has been applied to $A \wedge B$ in previous stages, it reduces to $\Gamma, A, B, A \wedge B \succ \Delta$; and if it has the form $\Gamma \succ A \wedge B, \Delta$ and no reduction has been applied to $A \wedge B$ in previous stages, the reduction tree branches into $\Gamma \succ A \wedge B, A, \Delta$ and $\Gamma \succ A \wedge B, B, \Delta$ and $\Gamma \succ A \wedge B, A, B, \Delta$. (iv) Similarly, $\Gamma \succ A \vee B, \Delta$ reduces to $\Gamma \succ A \vee B, A, B, \Delta$; and $\Gamma, A \vee B \succ \Delta$ reduces to $\Gamma, A, A \vee B \succ \Delta$ and $\Gamma, B, A \vee B \succ \Delta$ and $\Gamma, A, B, A \vee B \succ \Delta$. (v) $\Gamma, Tr(\bar{A}) \succ \Delta$ reduces to $\Gamma, A, Tr(\bar{A}) \succ \Delta$; and $\Gamma \succ Tr(\bar{A}), \Delta$ reduces to $\Gamma \succ Tr(\bar{A}), A, \Delta$.

Lemma 56. *The set of states deemed impossible by a sequent is the union of the states deemed impossible by the sequents to which it reduces in a reduction tree.*

Proof. We look at each clause in the reduction procedure. The lemma holds trivially for clause (i). It holds for (ii) because the truth-makers of $\neg A$ are exactly the falsity-makers of A, and vice versa. The other cases, in particular those for (v), are analogous except for when the reduction tree branches out, such as in the case of $\Gamma \succ A \wedge B, \Delta$. In this case, the lemma holds because the falsity-makers of $A \wedge B$ are the union of the falsity-makers of A, the falsity-makers of B, and any fusion of such falsity-makers, which corresponds to the three sequents that result from the reduction. ■

Definition 57 (Sequents resulting from an open branch of a reduction tree)**.** If an open branch of a reduction tree terminates, the resulting sequent is the leaf of that branch. If the open branch does not terminate, then the resulting sequent is the sequent $\Gamma_\omega \succ \Delta_\omega$, where Γ_ω is the union of all the sets on the left side of sequents in this open branch and and Δ_ω is the union of the sets on the right side of sequents in the branch.

Lemma 58. *Let* $\Gamma \succ \Delta$ *be a sequent resulting from an open branch, let* Γ^{at} *be the set of atomic sentences in* Γ*, and let* Δ^{at} *be the set of atomic sentences in* Δ*. Then a state that is deemed impossible by* $\Gamma^{at} \succ \Delta^{at}$ *includes as a part a truth-maker for every sentence in* Γ *and a falsity-maker for every sentence in* Δ*.*

Proof. We argue by induction on the complexity of sentences in $\Gamma \cup \Delta$. The states deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ trivially include truth-makers for every atomic sentence in Γ and falsity-makers for every atomic sentence in Δ. Suppose our lemma holds for sentences up to complexity n, and let us consider sentences of complexity $n + 1$. Note that since we have an open branch, we know that all possible reduction procedures have been applied. For negations in Γ, we know that the negatum, which is of complexity n, is in Δ. So, by our induction hypothesis states deemed impossible by $\Gamma^{at} \succ \Delta^{at}$ include a falsity-maker for the negatum, which is a truth-maker for our negation. Similarly for all other connectives where the reduction tree does not branch. For disjunctions on the left, we know that Γ contains also one or both of the disjunctions, which are of complexity n. So by our hypothesis, $\Gamma^{at} \succ \Delta^{at}$ contains truth-makers for one or both disjuncts, and any of these options ensures that it includes a truth-maker for the disjunction. Similarly for conjunctions on the right. ■

Proposition 59. $\Gamma \models_{\mathsf{STT}} \Delta$ *if and only if* $\Gamma \mathrel{\substack{STT \\ \models \\ LTM}} \Delta$.

Proof. Left-to-right: The proofs for Proposition 53 and Proposition 51 still go through, because clauses (tr+) and (tr−) ensure that [Lt] and [Rt] are valid for $\mathrel{\substack{STT \\ \models \\ LTM}}$.

Right-to-left: Suppose that there is no proof of $\Gamma \succ \Delta$. Hence, a reduction tree for $\Gamma \succ \Delta$ has an open branch. Let $\Gamma_\omega \succ \Delta_\omega$ be the sequent that results from that branch, and let $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ be the sequent that results from $\Gamma_\omega \succ \Delta_\omega$ by omitting all complex sentences. We can use as our desired countermodel any model that makes possible one of the states deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$. For by Lemma 58, any state that is deemed impossible by $\Gamma^{at}_\omega \succ \Delta^{at}_\omega$ is also deemed impossible by $\Gamma_\omega \succ \Delta_\omega$. And by Lemma 56, any state deemed impossible by $\Gamma_\omega \succ \Delta_\omega$ is deemed impossible by $\Gamma \succ \Delta$. So our model is a countermodel to $\Gamma \succ \Delta$. We know that there is such a model because any model will work that makes only those states impossible that are required to be impossible by Exclusivity, and $\Gamma^{at}_\omega \cap \Delta^{at}_\omega = \emptyset$. ■

Proposition 60. $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$ *if and only if* $\Gamma \mathrel{\substack{\mathfrak{B} \\ \models \\ TM}} \Delta$.

Proof. Left-to-right: By induction on proof-height. The axioms of $\mathsf{NMMS}_{\mathfrak{B}}$ are all the sequents in $\mathrel{\substack{\mid\!\sim \\ \mathfrak{B}}}$. Hence, they are valid according to $\mathrel{\substack{\mathfrak{B} \\ \models \\ TM}}$ by Definition 48. Our sequent rules preserve validity in TM-models by Proposition 51.

Right-to-left: Suppose that $\Gamma \succ \Delta$ is not derivable in $\mathsf{NMMS}_{\mathfrak{B}}$. A root-first proof-search must yield an atomic sequent that is not in $\mathrel{\substack{\mid\!\sim \\ \mathfrak{B}}}$. Hence, the states that this atomic sequent deems impossible are not all impossible in all models. So we can find a model in which one of these states is possible. By Proposition 52, this is also a counterexample to $\Gamma \mathrel{\substack{\mathfrak{B} \\ \models \\ TM}} \Delta$. ■

Notes

1 In the following chapters, we will look at contents in their own right, which will ultimately also allow us to think of reason relations among contents in abstraction from relations among sentences. But in this chapter, we take sets of sentences as the relata of reason relations.

2 The ideas in this chapter have been developed in conjunction with the ideas and results presented in (Hlobil, 2022a). That paper does not, however, include any treatment of nonmonotonic consequence relations. For some of the technical

results below, it might be helpful (though not necessary) to consult (Hlobil, 2022a) and (Hlobil, 2022b).

3 We think of the contentfulness of contentful items not as consisting in a relation between these items and other items that are their contents but rather as a property of contentful items. We talk about (worldly) propositions below, but those are not contents but contentful items. So, our notion of contentful item here includes what we mean by "proposition," as will become clearer below.

4 As in the previous chapters, we restrict ourselves to declarative sentences. So we will not consider whether and how questions or imperatives might be said to represent anything.

5 This formulation is, of course, inspired by Aristotle's explanation of truth and falsity: "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true" (Aristotle, Meta. Γ, 1011b25).

6 In contrast to some advocates of truth-maker theory, we avoid talk about states existing or not existing to mean something like that the state obtains or does not obtain. We think that such talk invites philosophical confusion whose resolution is beyond the scope of this book.

7 This distinction is analogous to the distinction between the actual truth-value of a sentence and its truth-value in each possible world, which is often called its "intension." However, truth-maker theory does not limit us to possible worlds, and it allows us to single out the parts of reality that are responsible for the truth or falsity of a given sentence.

8 Fine (2017a, 629) uses the term "bilateral proposition" for what we here call a "worldly proposition."

9 We are here again developing an Aristotelian idea. According to Aristotle, the intellect and what it thinks about share their form (as do sensation and the sensed object).

> Within the soul the faculties of knowledge and sensation are potentially these objects, the one what is knowable, the other what is sensible. They must be either the things themselves or their forms. The former alternative is of course impossible: it is not the stone which is present in the soul but its form. It follows that the soul is analogous to the hand; for as the hand is a tool of tools, so thought is the form of forms and sense the form of sensible things. (Aristotle, De An. 431b25-432a3)

In knowing an object, the faculty of knowledge takes on the form of the known object and, hence, becomes isomorphic to that object. The nature of the intellect is that it can take on forms in this way. That is why it is the form of forms. In the special case in which what is thought about does not have any matter the known object is actually in the intellect: "For in the case of objects which involve no matter, what thinks and what is thought are identical; for speculative knowledge and its object are identical" (Aristotle, De An. 430a4-6).

The Aristotelian doctrine that understanding and what is understood share their form was accepted by Aquinas. As Brower and Brower-Toland (2008) point out, an adequate interpretation of Aquinas's account—as well as Aristotle's—must tell us what the "intentional possession" of a form by the mind (or, for us, the possession of a form by a sentence) is. In particular, an interpretation of Aquinas as making a claim of "formal-sameness" must say how forms that are not forms of accidents can occur in the soul in such a way that their occurrence is an accident of the soul. In our context, only forms of something like facts (worldly propositions and sentences) matter; and it seems that facts can occur in the world and also in the mind or in discourse. We think that, in our context, "intentional possession" can be explained in terms of normative incompatibility. Hence, we think that we can avoid the problems that Brower and Brower-Toland identify for the view that they call the "Formal-Sameness Theory." A fuller discussion of such exegetical issues goes beyond the scope of this book.

10 This is the basic idea of information theoretic semantics (see Dretske, 1981). Note, however, that we are developing this idea in a very different direction, which becomes clearer in the next sentences.

11 The traditional notion of substantial form is, of course, intimately connected to many difficult philosophical and exegetical issues. We use the notion in a more metaphysical and abstract sense, and not in the sense of an inner efficient cause of certain properties (see Pasnau, 2004). However, we use the notion merely as an expository device; hence we can ignore the many philosophical and exegetical issues here.

12 Note that one state can be an occurrence of several rational forms, and there might be a way in which an assertion or denial can be an occurrence of several rational forms as well (although we do not discuss this possibility here). According to some theories of substantial forms (notably according to Aquinas's view), this cannot happen with substantial forms. So, this is a potential difference between rational forms and substantial forms.

13 This is again inspired by Aristotle. In the *Categories*, Aristotle notes that assertions and denials are incompatible with each other in a way that mirrors the way in which the states that they represent are incompatible:

> [W]hat underlies an affirmation or negation [is not] itself an affirmation or negation. For an affirmation is an affirmative statement and a negation a negative statement, whereas none of the things underlying an affirmation or negation is a statement. These are, however, said to be opposed to one another as affirmation and negation are; for in these cases, too, the manner of opposition is the same. For in the way an affirmation is opposed to a negation, for example "he is sitting"—"he is not sitting", so are opposed also the actual things underlying each, his sitting—his not sitting. (Aristotle, Cat. 12b5-b16)

We hold that assertion and denial (and acceptance and rejection)—which correspond to Aristotle's affirmation and negation—are opposed to each other normatively. The truth-makers and falsity-makers—which correspond to Aristotle's "things underlying an affirmation or negation"—are opposed to each other alethically. The "manner of opposition" is the same because the two opposition relations are isomorphic.

14 This terminology was introduced by Brandom (2019) in *A Spirit of Trust* (starting at p. 84). However, there Brandom analogizes content and matter,

which we do not do here. As will become clearer in the next chapter, we take conceptual contents to be rational forms.

15 Below, we will discuss whether it can happen that both a truth-maker and a falsity-maker obtain, and whether it can happen that neither obtains. The claim that the first is impossible turns out to be equivalent to the structural rule of Containment, and the claim that the second cannot occur turns out to be equivalent to the structural rule of Cut. Hence, the fact that this passage doesn't rule out either possibility is a reflection of the fact that we are interested in theorizing open reason relations.

16 Quine's inscrutability of reference, as well as Putnam's and Davidson's arguments that truth-conditions do not suffice to determine reference relations can be viewed as versions of this general issue.

17 Fine's notation for fusion is "$\sqcup$", which we will use to express something else in the next chapter. Hence we use "$\Cup$" to denote the fusion of states.

18 Thus the meanings of logical connectives are functions from some worldly propositions to a worldly proposition.

19 We do not foresee any particular problems with the extension of our results to first-order logic. However, there are well-known unresolved questions regarding the truth- and falsity-makers of universal generalizations (Fine 2017c, sec I.7; Jago 2018, chap 5). For example, the truth-maker of "All humans are mammals" cannot just be a fusion of the truth-makers of its actual instances because this fusion does not rule out the possibility that there is another human who is not a mammal. One option is to add totality-facts and to say that a truth-maker of a universal generalization is a fusion of truth-makers for all its instances and the totality-fact that these are all the instances (Armstrong, 2004, 19). Fine prefers a solution in terms of arbitrary objects (Fine, 2017c, 568-569). We suspect that any plausible treatment of these issues can be reproduced within our pragmatics-first approach, so that the isomorphism that is our topic here will continue to hold for the resulting first-order systems.

20 The semantic clause for the falsity-makers of conjunctions that we use here is sometimes called the "inclusive clause." The non-inclusive alternative clause that is sometimes used in truth-maker theory is the following:

(and$-_n$) $s \dashv\!\| B \wedge C$ if and only if $s \dashv\!\| B$ or $s \dashv\!\| C$

This non-inclusive clause allows that a fusion of falsity-makers for the two conjuncts is not an exact falsity-maker of the conjunction. Thus, the state of it neither raining nor being night is not necessarily a falsity-maker of "It is raining and it is night." By contrast, according to the inclusive clause for the falsifiers of conjunctions, a fusion of falsifiers for each conjunct is a falsifier for the conjunction. The dual non-inclusive clause for the truth-makers of disjunctions is the following:

(or$+_n$) $s \|\!\vdash B \vee C$ if and only if $s \|\!\vdash B$ or $s \|\!\vdash C$

As one can see when one considers the correspondence that we will explain below, the non-inclusive clauses correspond to the Ketonen-style sequent rules of $\mathsf{NMMS}^{\backslash ctr}$. If we adopt the non-inclusive clauses and allow Contraction to fail by defining fusions of states in such a way that fusion is no longer idempotent, that is, the state $s \Cup s$ is not always identical to the state s, then the resulting theory coincides with $\mathsf{NMMS}^{\backslash ctr}$ in its consequence relation, given the same base vocabulary. What this means will become clearer at the end of this chapter, but we will not work out the details. Rather, we will postpone a discussion of non-contractive versions of our theory until the next chapter.

21 We write $\bigwedge\{\gamma_1, ..., \gamma_k\}$ for $\gamma_1 \wedge ... \wedge \gamma_k$, and similarly $\bigvee\{\gamma_1, ..., \gamma_k\}$ for $\gamma_1 \vee ... \vee \gamma_k$.

22 A lot of the work of this assumption can be done by using canonical models (see Fine, 2017a, 647). This works, for instance, for completeness proofs. However, certain points can be seen more clearly with the assumption in place.

23 This formulation differs from Fine's in the quantification over further states u. In the presence of Downward-Closure, the two formulations are equivalent.

24 Stipulating these constraints for atomic sentences suffices (given the semantic clauses) to enforce them for the whole language.

25 Fine takes the relata of consequence to be propositions. We will work with sentences. Assumption 39 ensures that this is unproblematic for our purposes.

26 We will talk about validity in a model, so that the term "validity" has no connotation of a quantification over models in our context. We will later define notions of validity that involve quantification over models. However, this is not essential for the thus defined relations to be relations of validity. We usually use "validity" when we are thinking in model-theoretic terms, even when no quantification over models is involved.

27 The idea behind truth-maker consequence is similar to Bueno and Shalkowski's modalism about logic, which says that B follows from A if and only if the conjunction of A and the negation of B is impossible (Bueno and Shalkowski, 2013, 11-12). In contrast to modalism, however, truth-maker consequence is formulated at a meta-theoretic level.

28 At this point it is again convenient that Definition 38 ensures that we always have a state to evaluate, even if $\Gamma \cup \Delta = \emptyset$ or if Γ has no verifiers or Δ no falsifiers. In such cases the relevant state is $\blacksquare$.

29 Indeed, Fine tends to consider it an advantage of his notions of consequence that they don't require us to distinguish the possible from the impossible states.

30 Bob Hale (2020, 132) uses a similar technique in order to capture non-logical necessities in truth-maker theory. And Gil Sagi (2018) has developed a broadly similar idea to fix meanings by constraints on models outside of truth-maker theory. As will become clearer below, we can afford to restrict our models all the way down to a single model. For our models do not settle the truth-values of sentences. Our models are, in this respect, more similar to Kripke frames (without a designated actual world) than to familiar first-order models.

31 These correspondences are explained at a more technical level in other work (Hlobil, 2022a). Here we aim to keep the discussion as accessible as possible. The reader interested in the formal details is referred to this other work and to the appendix to this chapter.

32 Notice that if the norms that govern the use of a sentence are not isomorphic to the modal profile of the worldly proposition that the sentence is meant to represent, then this is not a case of misrepresentation. Rather it is a failure to represent what one means to represent, correctly or incorrectly. Of course, it is an idealization that our conceptual norms are indeed isomorphic to the alethic-modal structure of the worldly propositions that we take them to represent. Thus, we will ultimately have to make room for imperfect or approximate isomorphisms. In this book, however, we ignore such issues, which are closely related to the issue of how our conceptual norms can change rationally over time, and to Semantic Covariant Tracking.

33 See the Appendix for technical details.

34 We also saw that Containment corresponds to Exclusivity. Since rejecting Containment isn't of much importance for us, however, we will only mention it in passing. In the previous chapter, we also talked about Contraction. Giving up Contraction in truth-maker theory would require that we define fusion not as a least upper bound. We will not pursue this here, but we will generalize the insights from the current chapter in the next one, where the issue of Contraction comes up again.

35 This kind of semantics for STT was first developed and presented in (Hlobil, 2022a) and (Hlobil, 2022b). In these papers, there are hints at the greater power of truth-maker semantics relative to the usual strong Kleene semantics for STT. The treatment of nonmonotonic consequence relations in this framework that we are offering below makes good on the promissory notes in these papers.

36 We leave out quantifiers, thus restricting us, in effect, to pure predicate logic (that is, predicate logic without quantifiers or identity). The truth predicate and the name of the liar sentence are the only things we really need from the language of the predicate calculus.

37 This calculus is equivalent, but not identical, to standard formulations of STT in the literature (see Hlobil, 2022a). To see this, note that, as a sequent calculus, STT can be formulated by adding [Lt] and [Rt] to Gentzen's calculus LK. But we know that the narrowly logical part of NMMS is equivalent to Gentzen's LK. So, adding [Lt] and [Rt] to NMMS yields a formulation of STT.

38 Here, we stipulate means of self-reference by fiat: we let λ and $\neg Tr(\bar{\lambda})$ simply be identical. This allows us to avoid the complications of adding self-reference via Gödel numbers. See Ripley (2012, 355) for more details.

39 We will be brief here, as the paradoxes are not really our topic. Readers interested in the technical details are referred to the cited paper.

40 There are well-known relations between ST and the logics LP, K3, and TS (Dicher and Paoli, 2019; Barrio et al., 2015). These relations can be spelled-out in truth-maker theory (see Hlobil, 2022a). If we do that, LP emerges roughly as the logic of impossible falsity-makers. K3 emerges roughly as the logic of possible truth-makers. And TS emerges as the logic that rejects Exclusivity. We will return to these connections in a more general setting in the next chapter.